

A background image of a sunset over sand dunes. The sky transitions from a deep blue on the left to a bright orange and yellow on the right. Two small silhouettes of people are visible on a ridge in the distance. The foreground shows the undulating curves of the sand dunes.

Diagrama de Bode

Sinais & Sistemas I
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Tópicos previstos

- ▶ Intro: o que é Resposta em Frequência;
- ▶ Como graficar resposta em frequência;
- ▶ Usando Matlab para obter diagramas de Bode;
- ▶ Como usar resposta em frequência para analisar resposta transitória de um sistema;

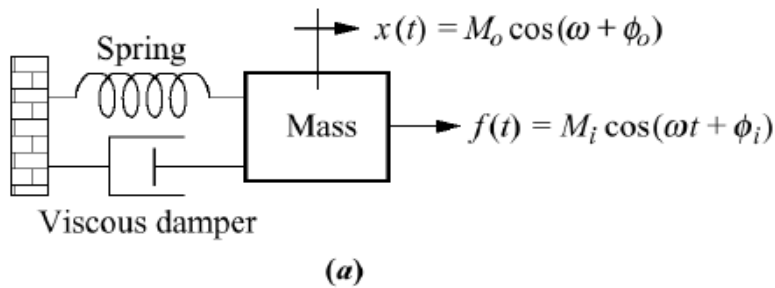


Introdução

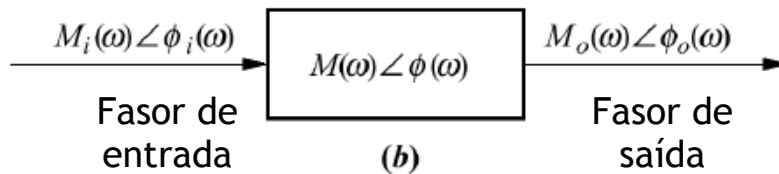
- ▶ Estabilidade e projeto da resposta transitória mediante ajuste de ganho:
 - ▶ Métodos baseados em resposta em frequência, diferentes do método baseado em RL, podem ser realizados sem a obrigatoriedade de uma ferramenta computacional usando aproximações assintóticas.
- ▶ O projeto da resposta transitória mediante compensação em cascata:
 - ▶ Métodos baseados em resposta em frequência não são tão intuitivos como os baseados em RL.
- ▶ Projeto dos erros de estado estacionário mediante compensação em cascata:
 - ▶ Métodos baseados em resposta em frequência facilitam o projeto de compensadores derivativos de forma a acelerar a resposta do sistema ao mesmo tempo respeitando requerimentos de erros de regime permanente.

Resposta em frequência: Definição...

- ▶ Ondas sinusoidais podem ser representadas como números complexos chamados fasores



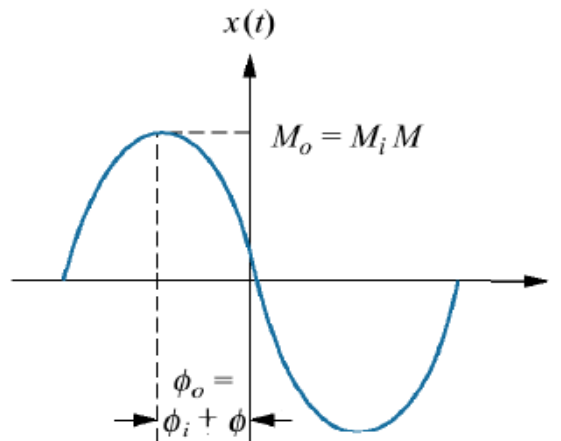
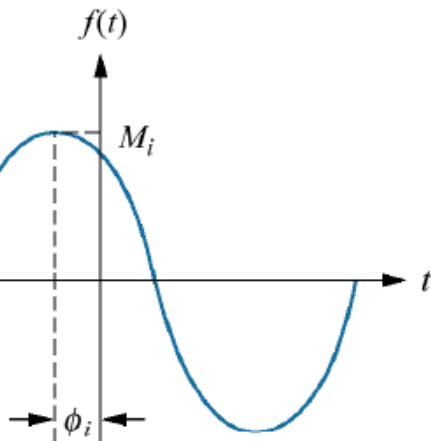
$f(t)$ = entrada de força, sinusoidal neste caso:



Definição:

$$M(\omega) = \frac{M_o(\omega)}{M_i(\omega)}$$

$$\phi(\omega) = \phi_o(\omega) - \phi_i(\omega)$$



Definições Matemáticas...

Relações de Euler:

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$z = x + jy = |z| [\cos(\varphi) + j \sin(\varphi)] = r e^{j\varphi}$$

x = Parte real de z ;

y = Parte Imaginária de z ;

$$r = |z| = \sqrt{x^2 + y^2} = \text{magnitude de } z;$$

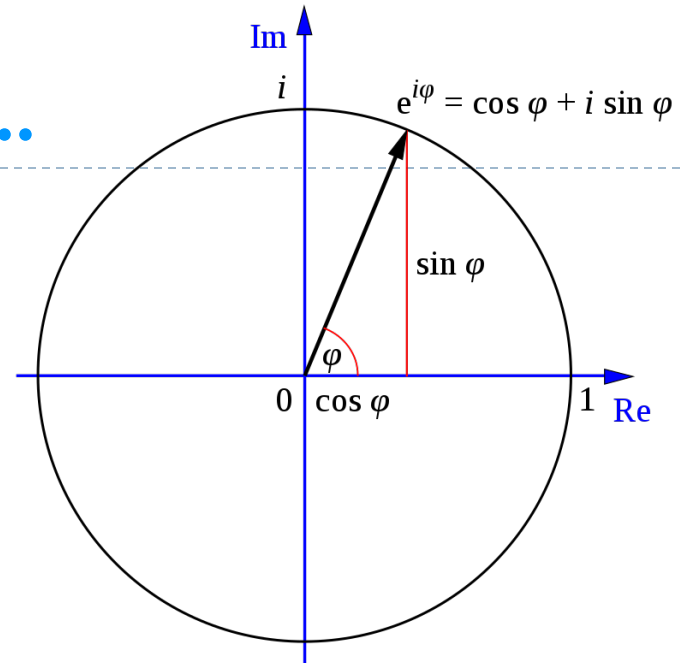
$$\varphi = \text{atan2}(y, x)$$

- ▶ Definição: $M \angle \phi(w)$
 - ▶ Magnitude: $M(w) = M_o(w) / M_i(w)$
 - ▶ Fase: $\phi(w) = \phi_o(w) - \phi_i(w)$

- ▶ Formatos de expressão:

$$r(t) = A \cos(\omega t) + B \sin(\omega t) = \sqrt{A^2 + B^2} \cos[\omega t - \tan^{-1}(B/A)]$$

1. Forma Polar: $M_i \angle \phi_i$
2. Forma Retangular: $A - jB$
3. Equação de Euler: $M_i e^{j\phi_i}$



Expressões analíticas p/ resposta em frequência

Seja o sistema mostrado na fig. ao lado.

Este sistema é excitado por uma entrada sinusoidal:

$$r(t) = A \cos(\omega t) + B \sin(\omega t)$$

$$r(t) = \sqrt{A^2 + B^2} \cos[\omega t - \tan^{-1}(B/A)]$$

$$r(t) = M_i \angle \phi_i = M_i e^{j\phi_i}, \text{ onde: } M_i = \sqrt{A^2 + B^2} \text{ e } \phi_i = -\tan^{-1}(B/A)$$

A resposta forçada do sistema $G(s)$ para esta entrada resulta em:

$$C(s) = \frac{As + B\omega}{(s^2 + \omega^2)} \cdot G(s)$$

Expandindo usando frações parciais, teremos:

$$C(s) = \frac{As + B\omega}{(s + j\omega)(s - j\omega)} = \frac{K_1}{s + j\omega} + \frac{K_2}{s - j\omega} + \text{Termos frações parciais de } G(s)$$

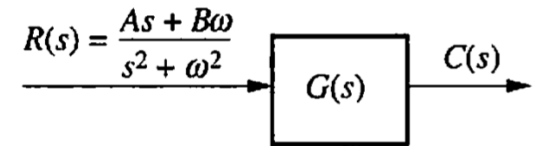
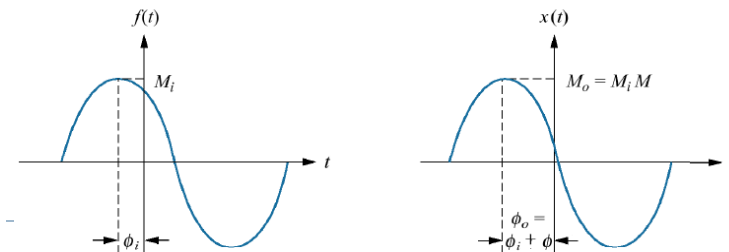
$$K_1 = \frac{As + B\omega}{s - j\omega} G(s) \Big|_{s \rightarrow -j\omega} = \frac{1}{2}(A + jB)G(-j\omega) = \frac{1}{2}M_i e^{-j\phi_i} M_G e^{-j\phi_G}$$

$$K_1 = \frac{M_i M_G}{2} e^{-j(\phi_i + \phi_G)}$$

$$K_2 = \frac{As + B\omega}{s + j\omega} G(s) \Big|_{s \rightarrow j\omega} = \frac{1}{2}(A - jB)G(j\omega) = \frac{1}{2}M_i e^{j\phi_i} M_G e^{j\phi_G}$$

$$K_2 = \frac{M_i M_G}{2} e^{j(\phi_i + \phi_G)} = K_1^*$$

onde: $M_G = |G(j\omega)|$ e $\phi_G = \angle G(j\omega)$.



Então $C(s)$ resulta:

$$C(s) = \frac{\frac{M_i M_G}{2} e^{-j(\phi_i + \phi_G)}}{s + j\omega} + \frac{\frac{M_i M_G}{2} e^{j(\phi_i + \phi_G)}}{s - j\omega}$$

Cuja transformada inversa de Laplace rende:

$$c(t) = M_i M_G \left(\frac{e^{-j(\omega t + \phi_i + \phi_G)} + e^{j(\omega t + \phi_i + \phi_G)}}{2} \right)$$

$$c(t) = M_i M_G \cos(\omega t + \phi_i + \phi_G)$$

$$c(t) = M_o \angle \phi_o = (M_i \angle \phi_i) \cdot (M_G \angle \phi_G)$$

Note que:

$M_G \angle \phi_G =$ resposta em frequência.

Ou seja:

$$G(j\omega) = G(s) \Big|_{s \rightarrow j\omega}$$

Exemplo 1: $G(s) = \frac{1}{s + 2}$

Como: $s = j\omega$, para obter: $G(j\omega) = G(s)|_{s \rightarrow j\omega}$

$$G(j\omega) = \frac{1}{j\omega + 2} = \frac{1}{2 + j\omega}$$

$$G(j\omega) = \frac{1}{j\omega + 2} = \frac{1}{2 + j\omega} \cdot \frac{2 - j\omega}{2 - j\omega} = \frac{2 - j\omega}{4 + \omega^2}$$

$$\text{Magnitude: } |G(j\omega)| = \frac{1}{|2 + j\omega|} = \frac{1}{\sqrt{2^2 + \omega^2}} = \frac{1}{\sqrt{\omega^2 + 4}}$$

$$\text{Magnitude(dB): } |G(j\omega)| \Big|_{dB} = 20 \log \left(\frac{1}{\sqrt{\omega^2 + 4}} \right)$$

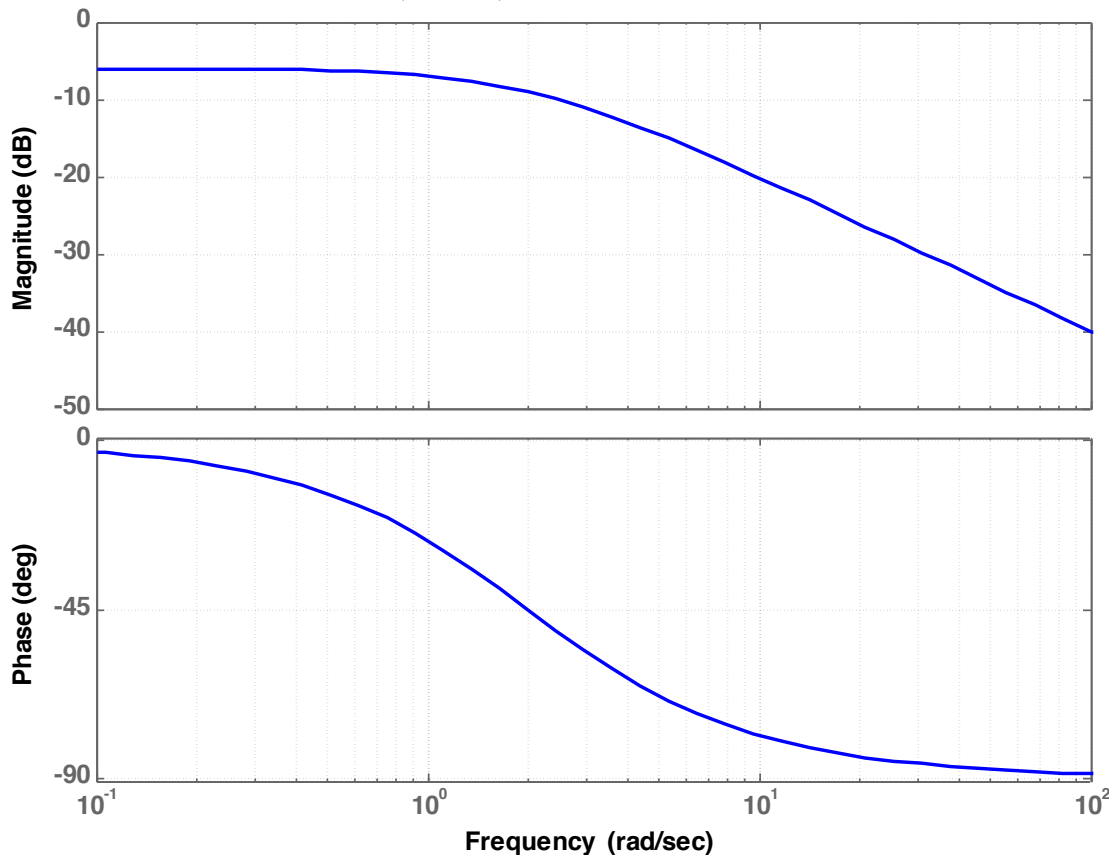
$$\text{Fase: } \angle G(j\omega) = \frac{1}{\tan^{-1}(\omega/2)} = -\tan^{-1} \left(\frac{\omega}{2} \right)$$

Exemplo 1: $G(s) = \frac{1}{s + 2}$

Grafico de Magnitude: $= 20 \log(1 / \sqrt{\omega^2 + 4})$

Grafico de Fase: $= -\tan^{-1}(\omega/2)$

Bode Diagram



```
>> clear all
>> numg=1;
>> deng=[1 2];
>> g=tf(numg,deng);
>> zpk(g)
```

Zero/pole/gain:

1

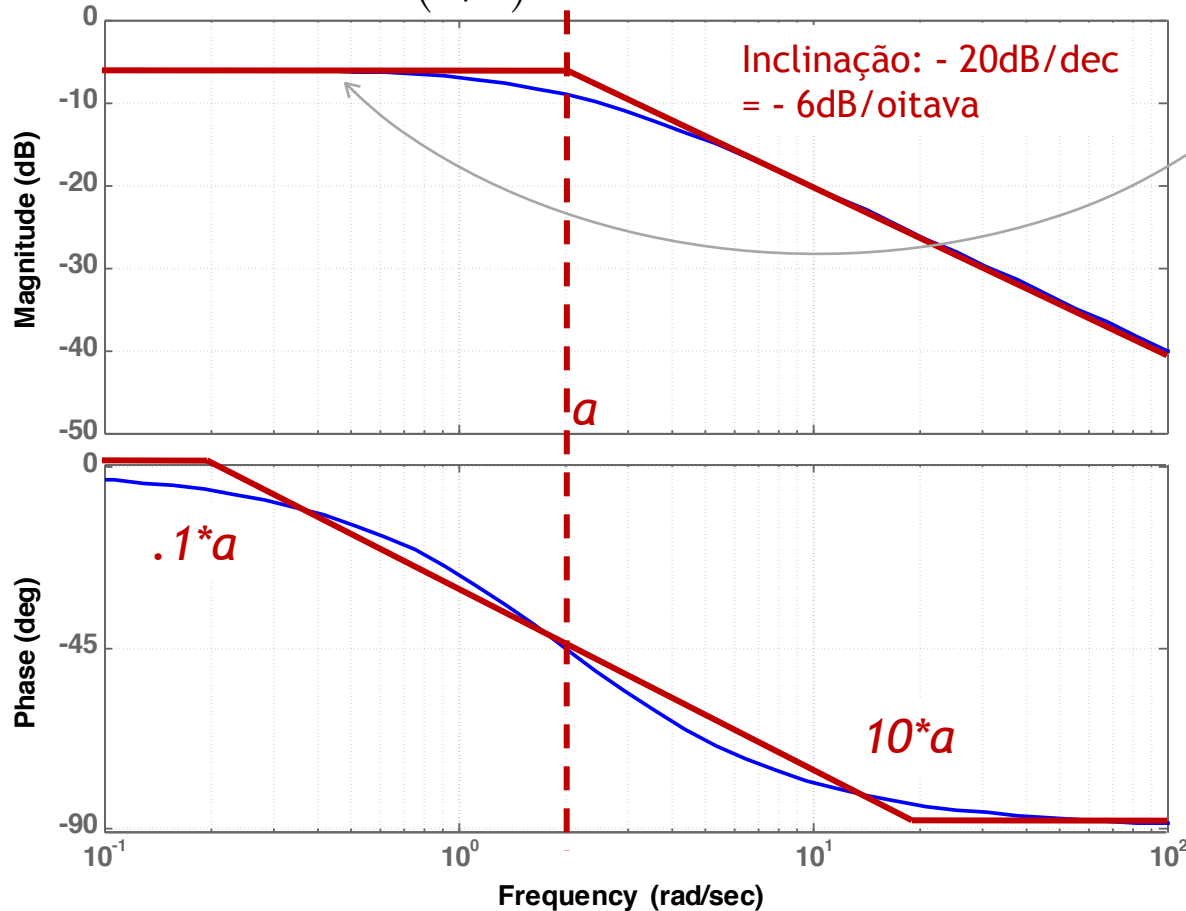
(s+2)

```
>> bode(g), grid
```


Exemplo 1: $G(s) = \frac{1}{s + 2}$

Gráfico de Magnitude: $= 20 \log(1/\sqrt{\omega^2 + 4})$ ou $= 20 \log(1/\sqrt{\omega^2 + a^2})$

Gráfico de Fase: $= -\tan^{-1}(\omega/2)$ **Bode Diagram**



Assintoticamente:

$$G(s) = \frac{1}{(s+a)} = \frac{1}{a\left(\frac{s}{a}+1\right)}$$

Para baixas freq. ($j\omega < a$; suponha $\omega = a/10$):

$$|G(j\omega)| = 20 \log\left(\frac{1}{\sqrt{\left(\frac{a}{10}\right)^2 + a^2}}\right) = 20 \log\left(\frac{1}{\sqrt{\frac{101a^2}{100}}}\right)$$

$$|G(j\omega)| \cong 20 \log\left(\frac{1}{a}\right) = -20 \log(a)$$

$$|G(j\omega)| = -20 \log(2) = -6,0205999133 \text{ dB}$$

Lembrando que:

$$\log_a b = x \quad \therefore \quad a^x = b$$

$$\log(1) = 0$$

$$\log(2) = 0,3010299957$$

$$\log(10) = 1$$

$$\log(100) = 2 \quad \therefore \quad 10^2 = 100$$

$$\log(1/10) = -\log(10) = -1$$

$$\log(a \cdot b) = \log(a) + \log(b)$$

$$\log(a/b) = \log(a) - \log(b)$$

Ref: URL: <http://www.swarthmore.edu/NatSci/echeeve1/Ref/LPSA/Bode/BodeHow.html>

Exemplo 1: $G(s) = \frac{1}{s + 2}$

Assintoticamente:

$$G(s) = \frac{1}{(s+a)} = \frac{1}{a\left(\frac{s}{a} + 1\right)}$$

Para baixas freq. ($j\omega < a$; suponha $\omega = a/10$):

$$|G(j\omega)| = 20 \log \left(\frac{1}{\sqrt{\left(\frac{a}{10}\right)^2 + a^2}} \right) = 20 \log \left(\frac{1}{\sqrt{\frac{101a^2}{100}}} \right)$$

$$|G(j\omega)| \cong 20 \log \left(\frac{1}{a} \right) = -20 \log(a)$$

$$|G(j\omega)| = -20 \log(2) = -6,0205999133 \text{ dB}$$

Para altas freq. ($j\omega > a$; suponha $\omega = 10a$):

$$|G(j\omega)| = 20 \log \left(\frac{1}{\sqrt{(10a)^2 + a^2}} \right) = 20 \log \left(\frac{1}{\sqrt{101a^2}} \right)$$

$$|G(j\omega)| \cong 20 \log \left(\frac{1}{\sqrt{100a}} \right) \cong 20 \log(1) - 20 \log(10a)$$

$$|G(j\omega)| \cong -20 \log(\omega)$$

$$|G(j\omega)|_{\omega=10a} \cong -20 \log(2 \cdot 10) = -26,0205999133 \text{ dB}$$

E para $\omega = a$, teremos:

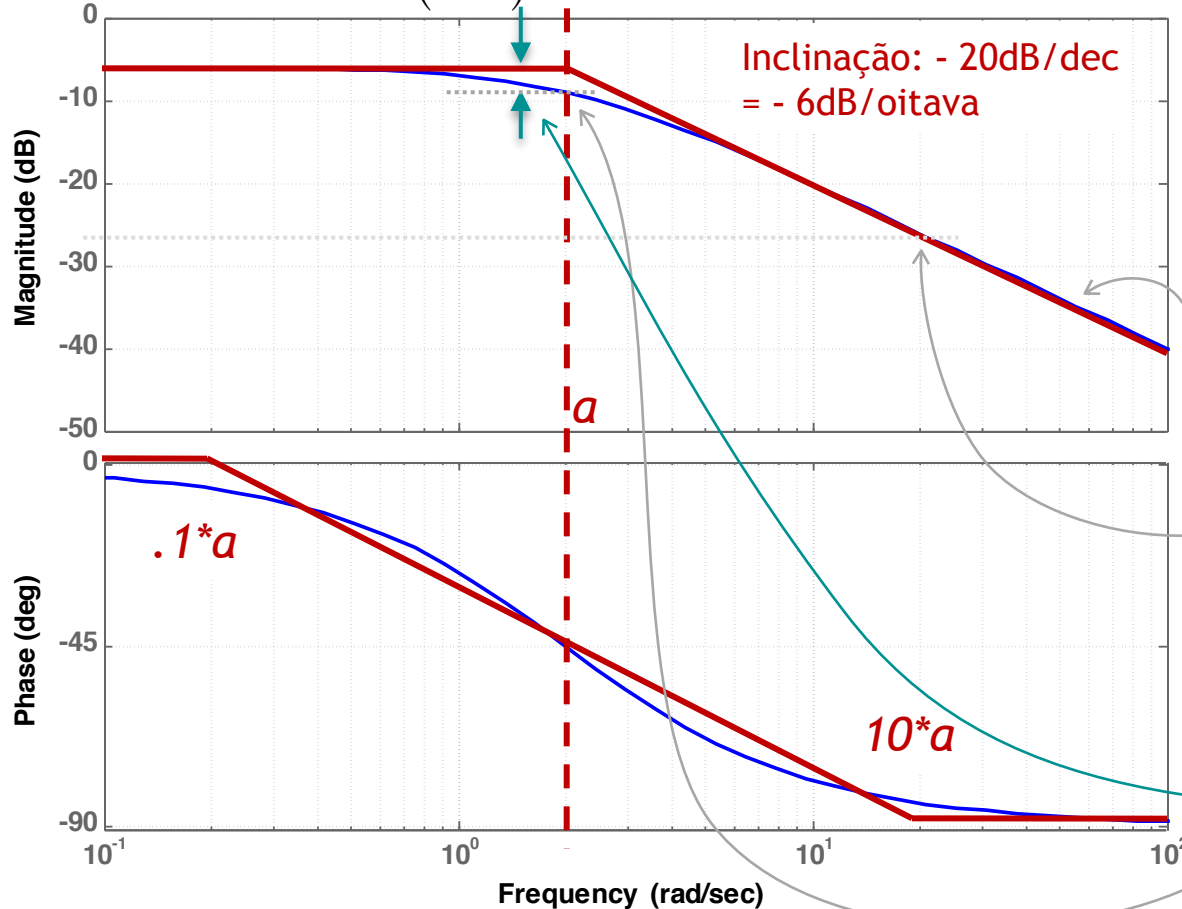
$$|G(j\omega)| = 20 \log \left(\frac{1}{\sqrt{a^2 + a^2}} \right)$$

$$|G(j\omega)| = -20 \log(\sqrt{2}a) = \underbrace{-20 \log(\sqrt{2})}_{-3,0103 \text{ dB}} - 20 \log a$$

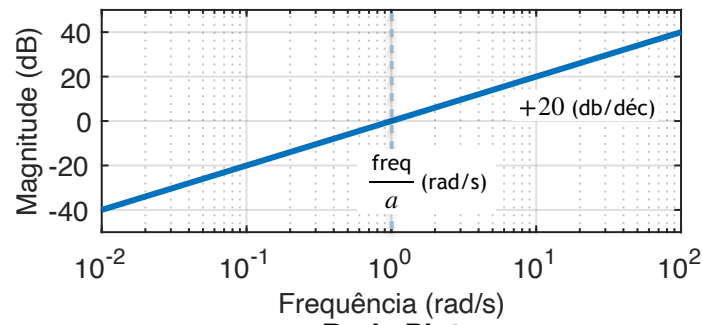
$$\underbrace{-20 \log(\sqrt{2})}_{-3,0103 \text{ dB}} - 20 \log 2 = -9,0309 \text{ dB}$$

Gráfico de Magnitude: $= 20 \log(1/\sqrt{\omega^2 + 4})$ ou $= 20 \log(1/\sqrt{\omega^2 + a^2})$

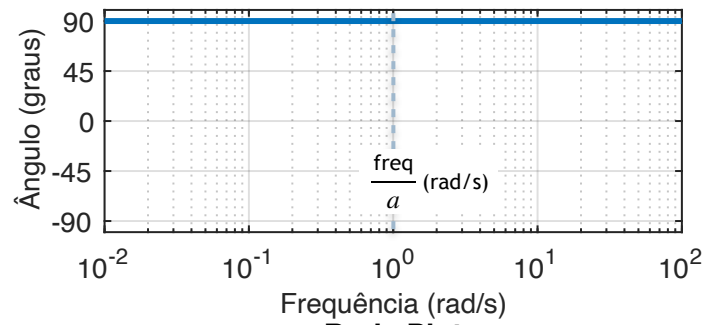
Gráfico de Fase: $= -\tan^{-1}(\omega/2)$ Bode Diagram



Bode Plot

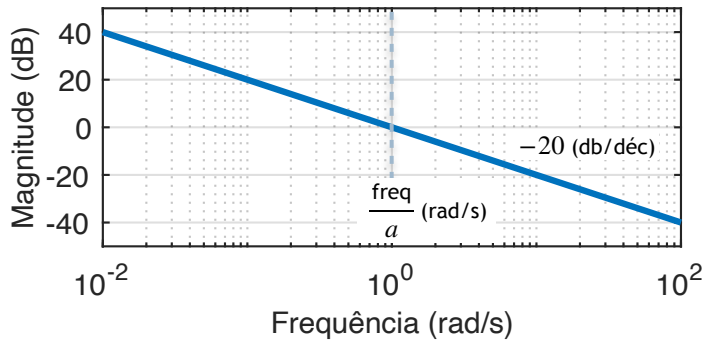


Bode Plot

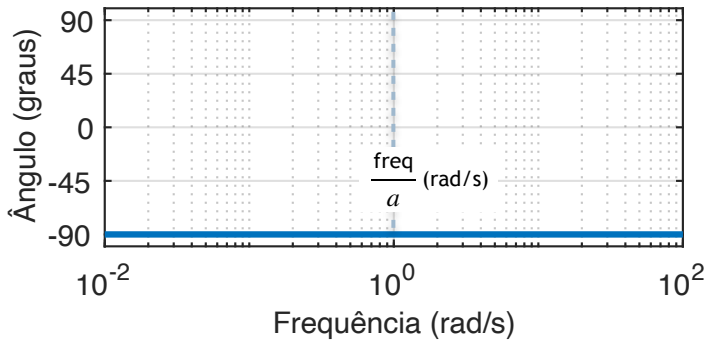


a) $G(s) = s$
(Derivador Puro);

Bode Plot

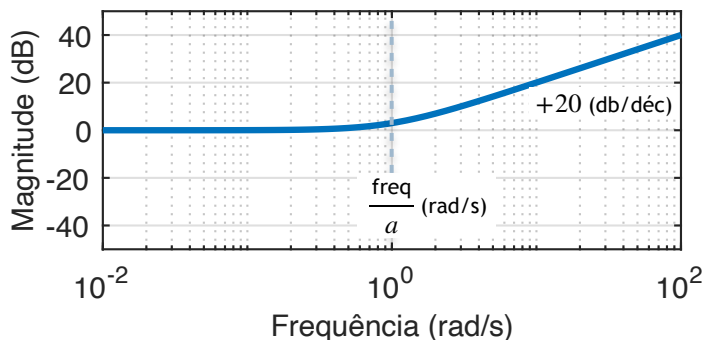


Bode Plot

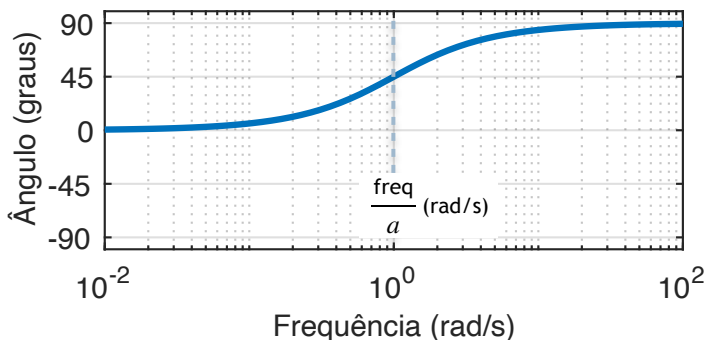


b) $G(s) = \frac{1}{s}$
(Integrador Puro);

Bode Plot

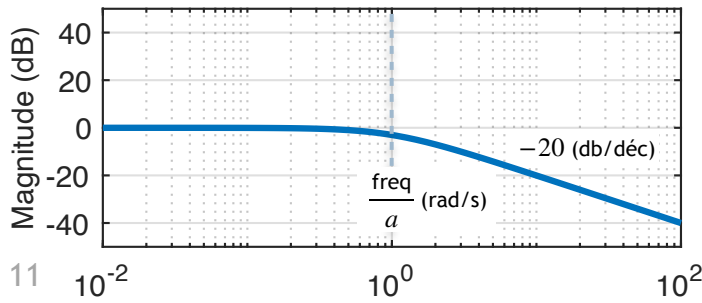


Bode Plot

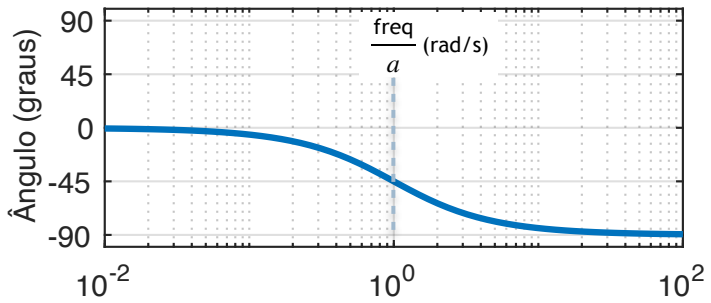


c) $G(s) = (s + a)$

Bode Plot



Bode Plot



d) $G(s) = \frac{1}{(s + a)}$

bode_lado_lado.m

```
% Mostrar diagrama de Bode: Magnitude x Fase, lado a lado
% Entrada: variável G = tf(.)
% Fernando Passold, em 10.06.2022
```

```
W={0.01,100};
[MAG,PHASE,Wb] = bode(G,W);
```

```
% Diagrama de Magnitude
subplot(121);
% bodemag(G,W); % até poderia ser usado
MAG=squeeze(MAG); % reduce dimensions
mag = 20*log10(MAG);
semilogx(Wb,mag,'LineWidth',3)
grid on
axis([0.01 100 -50 50])
title ('Bode Plot')
xlabel('Frequência (rad/s)');
ylabel('Magnitude (dB)');
yticks([-40 -20 0 20 40]);
```

```
% Diagrama de Fase
subplot(122);
% Detalhe: PHASE = 1 x 1 x 41 !!!
PHASE=squeeze(PHASE); % reduce dimensions
semilogx(Wb,PHASE,'LineWidth',3)
axis([0.01 100 -100 100])
grid on
title ('Bode Plot')
xlabel('Frequência (rad/s)');
ylabel('Ângulo (graus)');
yticks([-90 -45 0 45 90]);
```


Revisão de traçados de Diagramas de Bode...

▶ Diagrama de Bode para:

- ▶ Se $s = jw$: (Derivador Puro)

$$G(jw) = (jw + a) = a \left(j \frac{w}{a} + 1 \right)$$

- ▶ Para baixas frequências ($w < a$):

$$G(jw) \approx a$$

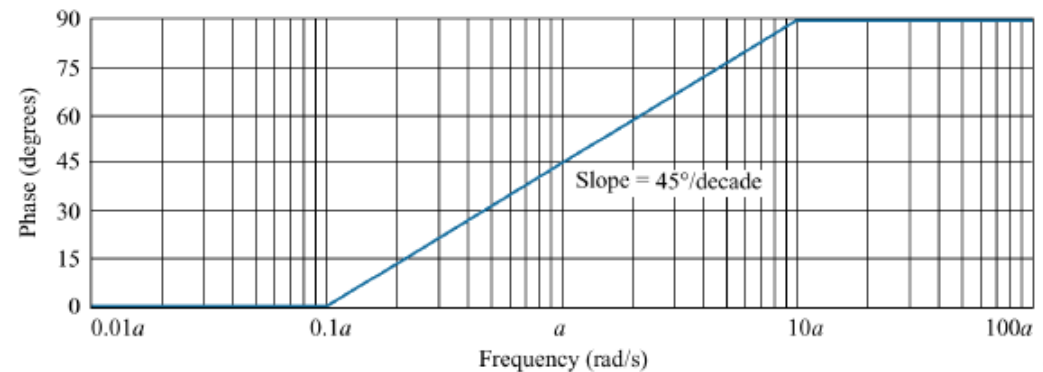
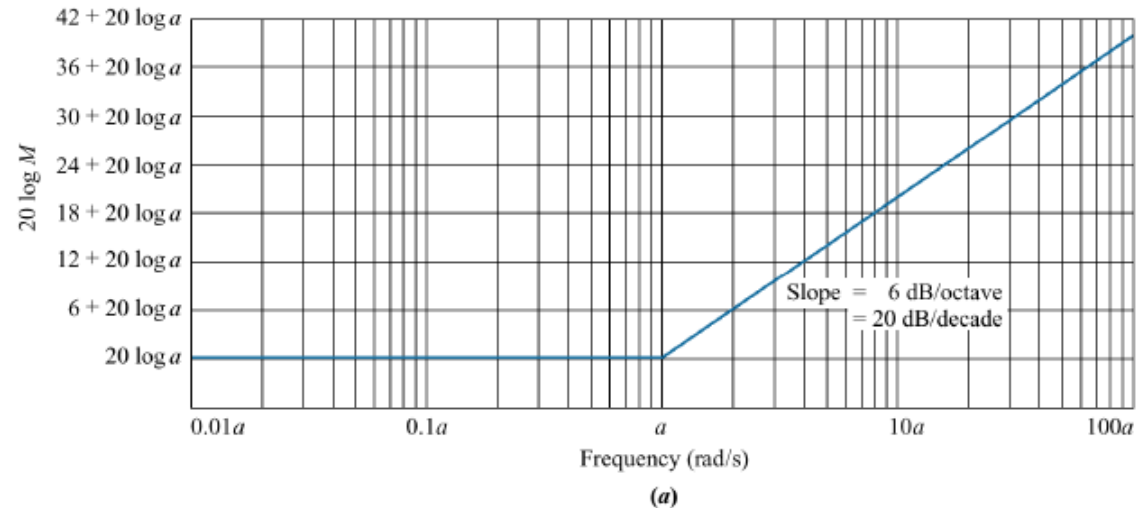
$$20 \log(M) = 20 \log(a)$$

- ▶ Para frequências elevadas ($w > a$):

$$G(jw) \approx a$$

$$20 \log(M) = 20 \log(a)$$

$$G(s) = (s + a)$$



Exemplo 2) $G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$

1. Cálculo do ganho “DC” (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Neste caso: Ganho DC quando aplicado degrau

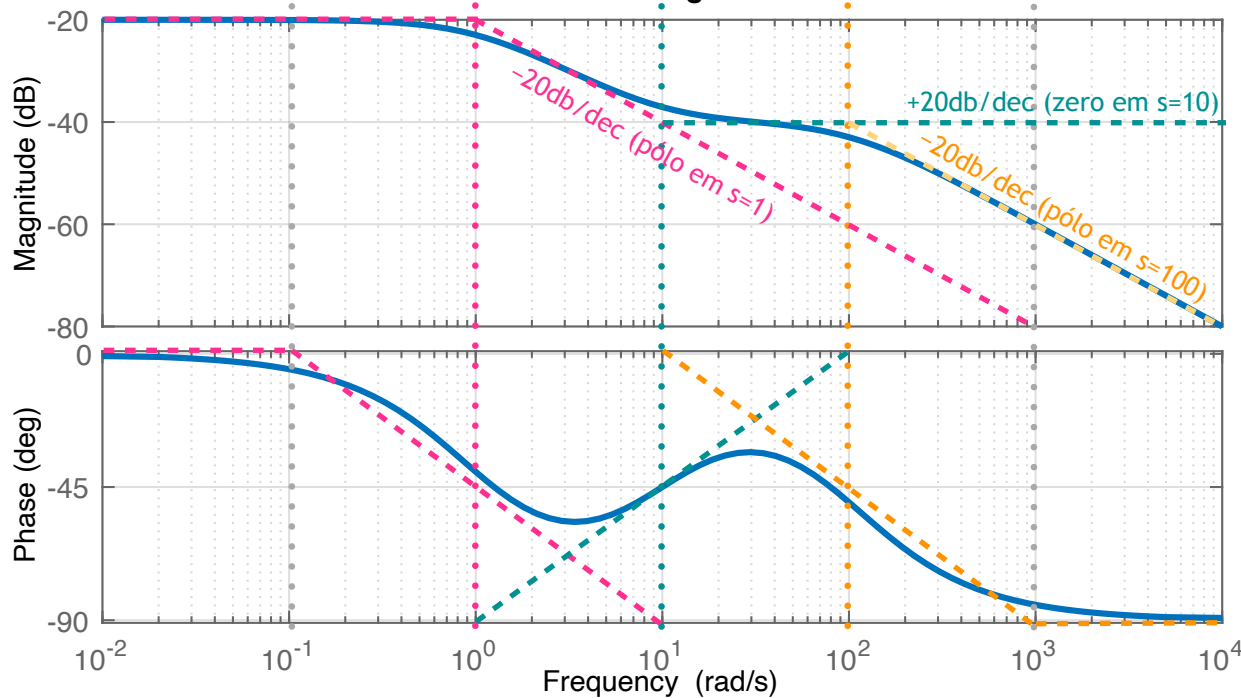
$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s+10)}{(s+1)(s+100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

Ganho DC $\left|_{\text{dB}}^{\text{Degrau}} = 20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$

2. Lembrar que:

cada pólo decreta ganho de: -20 db/déc;
 cada zero incrementa ganho de: +20 db/déc.

Bode Diagram



```
>> G=tf([1 10],poly([-1 -100]))
G =
```

$$\frac{s + 10}{s^2 + 101s + 100}$$

```
>> zpk(G)
```

$$\frac{(s+10)}{(s+100)(s+1)}$$

```
>> bode(G)
```

```
>> grid
```

Exemplo 2) $G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$

1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

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$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s+10)}{(s+1)(s+100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

$$\text{Ganho DC} \Big|_{\text{dB}} = 20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$$

Valores em $\omega = 0,1 \text{ rad/s}$

(1 década abaixo do pólo em $s = -1$):

$$G(s) = \frac{s + 10}{(s + 1)(s + 100)} = \frac{s + 10}{s^2 + 101s + 100}$$

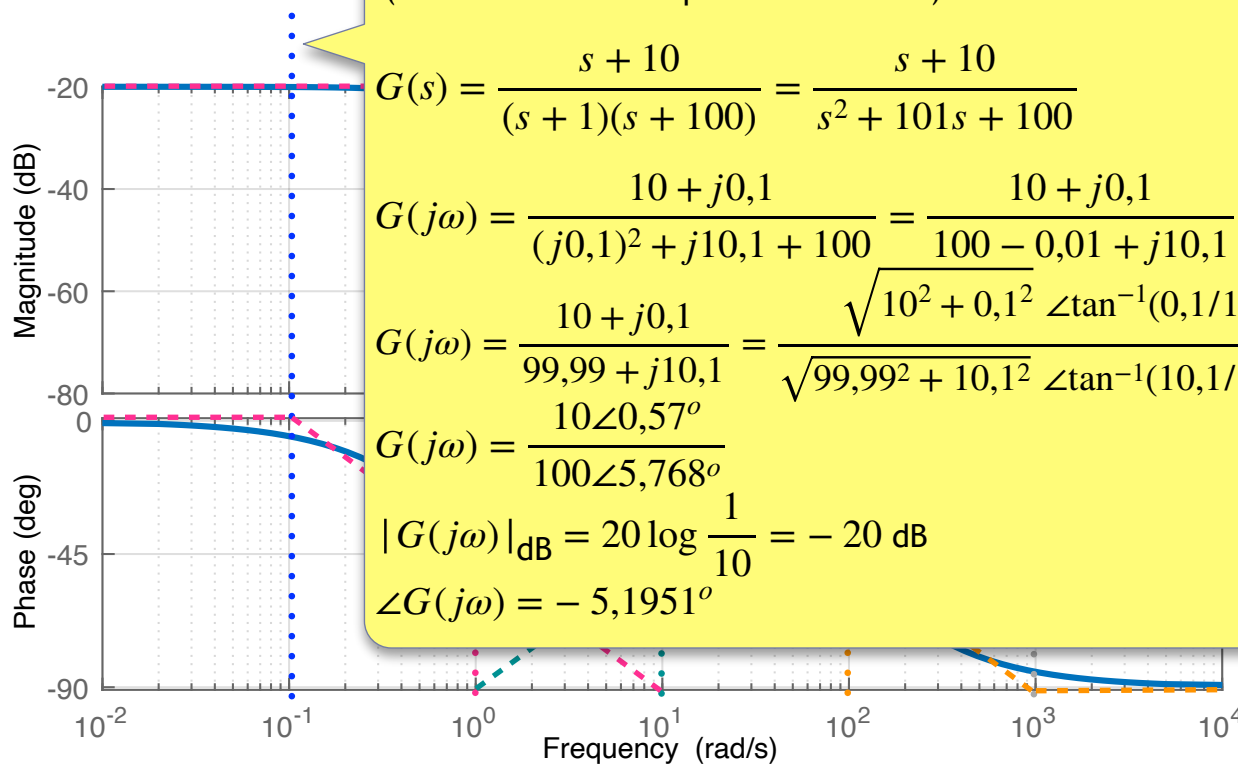
$$G(j\omega) = \frac{10 + j0,1}{(j0,1)^2 + j10,1 + 100} = \frac{10 + j0,1}{100 - 0,01 + j10,1}$$

$$G(j\omega) = \frac{10 + j0,1}{99,99 + j10,1} = \frac{\sqrt{10^2 + 0,1^2} \angle \tan^{-1}(0,1/10)}{\sqrt{99,99^2 + 10,1^2} \angle \tan^{-1}(10,1/99,99)}$$

$$G(j\omega) = \frac{10 \angle 0,57^\circ}{100 \angle 5,768^\circ}$$

$$|G(j\omega)|_{\text{dB}} = 20 \log \frac{1}{10} = -20 \text{ dB}$$

$$\angle G(j\omega) = -5,1951^\circ$$



de: -20 db/déc;
de: +20 db/déc.

```
>> G=tf([1 10],poly([-1 -100]))
G =
      s + 10
-----
s^2 + 101 s + 100
>> zpk(G)
      (s+10)
-----
(s+100) (s+1)
>> bode(G)
>> grid
```

Exemplo 2) $G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$

1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Neste caso: Ganho DC quando aplicado degrau

$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s + 10)}{(s + 1)(s + 100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

Ganho DC $\Big|_{\text{dB}} = 20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$

Valores em $\omega = 1 \text{ rad/s}$ (no pólo em $s = -1$):

$$G(s) = \frac{s + 10}{(s + 1)(s + 100)} = \frac{s + 10}{s^2 + 101s + 100}$$

$$G(j\omega) = \frac{10 + j1}{(j)^2 + j101 + 100} = \frac{10 + j1}{100 - 1 + j101}$$

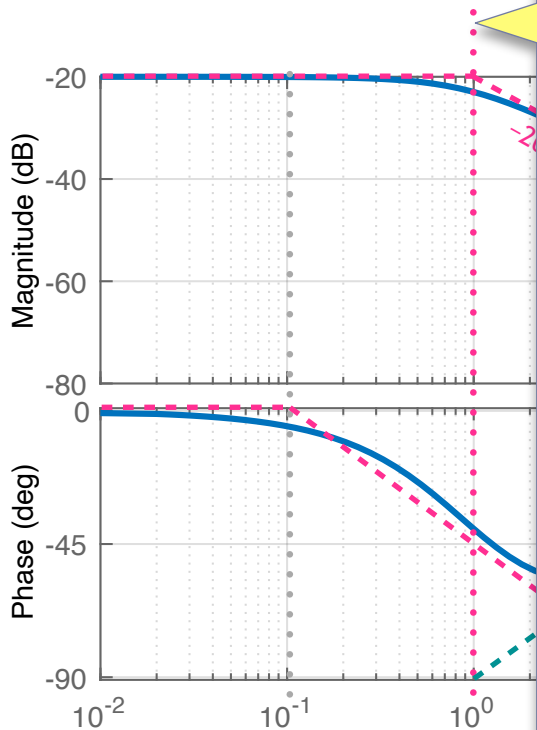
$$G(j\omega) = \frac{10 + j1}{99 + j101} = \frac{\sqrt{10^2 + 1^2} \angle \tan^{-1}(1/10)}{\sqrt{99^2 + 101^2} \angle \tan^{-1}(101/99)}$$

$$G(j\omega) = \frac{\sqrt{101} \angle 5.7106^\circ}{\sqrt{20002} \angle 45.573^\circ}$$

$$G(j\omega) = \frac{10,05 \angle 5,7106^\circ}{141,43 \angle 45,573^\circ} = 0,07106 \angle -39,862^\circ$$

$$|G(j\omega)|_{\text{dB}} = 20 \log(0,07106) = -22,968 \text{ dB}$$

$$\angle G(j\omega) = -39,862^\circ$$



déc;
déc.

poly([-1 -100])

100

Exemplo 2) $G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$

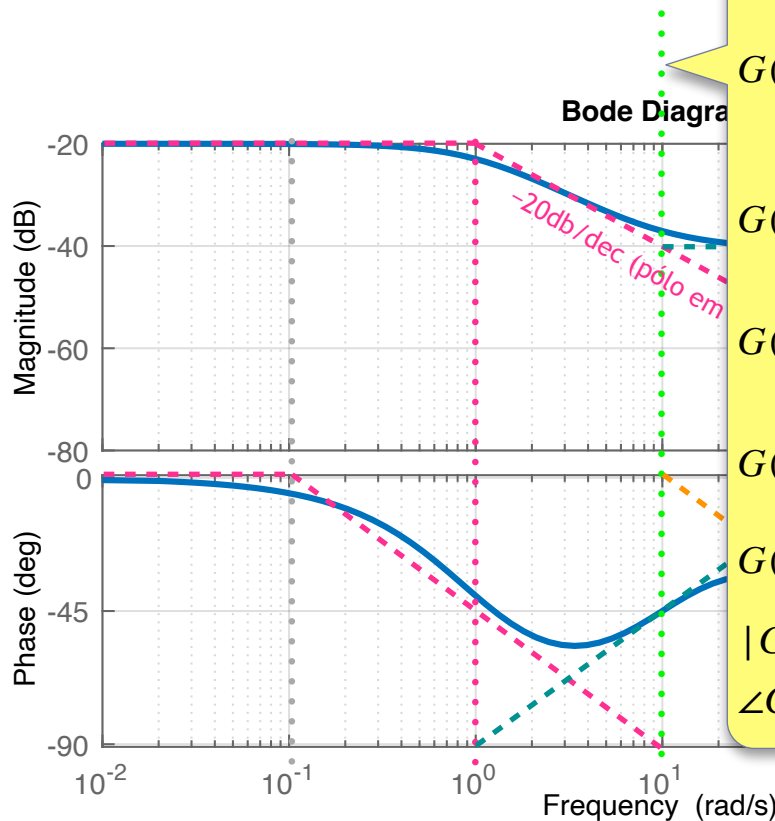
1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Neste caso: Ganho DC quando aplicado degrau

$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s+10)}{(s+1)(s+100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

Ganho DC = $20 \log(K \cdot 1/10) = -20 + 20 \log(K)$ dB



Valores em $\omega = 10$ rad/s (no pólo em $s = -10$):

$$G(s) = \frac{s + 10}{(s + 1)(s + 100)} = \frac{s + 10}{s^2 + 101s + 100}$$

$$G(j\omega) = \frac{10 + j10}{(j10)^2 + j1010 + 100} = \frac{10 + j10}{100 - 100 + j1010}$$

$$G(j\omega) = \frac{10 + j10}{j1010} = \frac{\sqrt{10^2 + 10^2} \angle \tan^{-1}(10/10)}{1010 \angle 90^\circ}$$

$$G(j\omega) = \frac{\sqrt{200} \angle 45^\circ}{1010 \angle 90^\circ}$$

$$G(j\omega) = \frac{14,142 \angle 45^\circ}{1010 \angle 90^\circ} = 0,014002 \angle -45^\circ$$

$$|G(j\omega)|_{dB} = 20 \log(0,014002) = -37,076 \text{ dB}$$

$$\angle G(j\omega) = -45^\circ$$

>> bode(G)
>> grid

Exemplo 2) $G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$

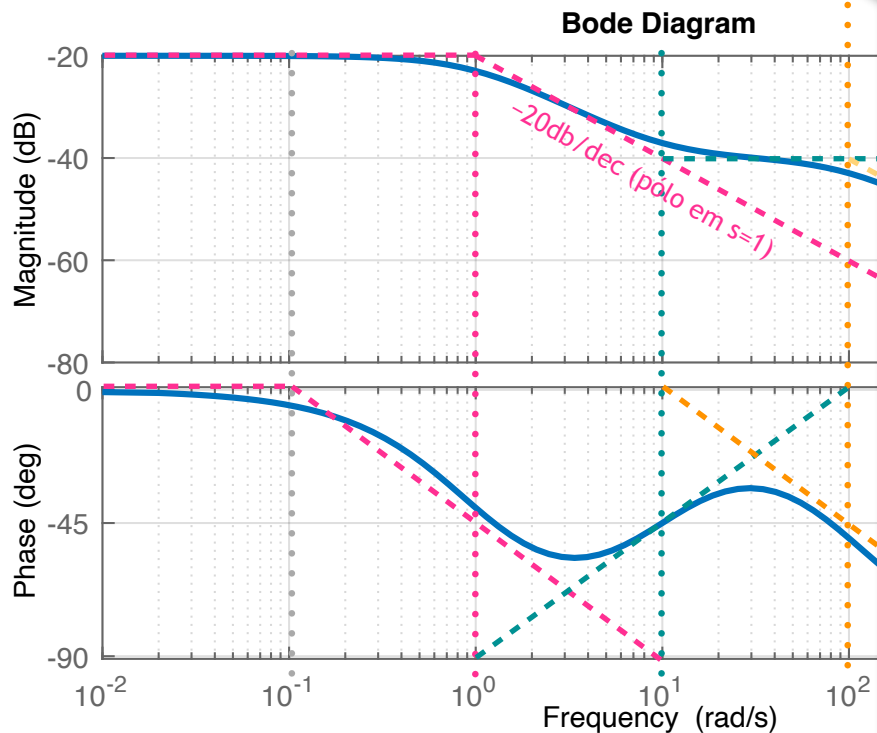
1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Neste caso: Ganho DC quando aplicado degrau

$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s+10)}{(s+1)(s+100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

Ganho DC = $20 \log(K \cdot 1/10) = -20 + 20 \log(K)$ dB



Valores em $\omega = 100$ rad/s (no pólo em $s = -100$):

$$G(s) = \frac{s + 10}{(s + 1)(s + 100)} = \frac{s + 10}{s^2 + 101s + 100}$$

$$G(j\omega) = \frac{10 + j100}{(j100)^2 + j10100 + 100} = \frac{10 + j100}{100 - 10000 + j10100}$$

$$G(j\omega) = \frac{10 + j100}{-9900 + j10100} = \frac{\sqrt{10^2 + 100^2} \angle \tan^{-1}(100/10)}{\sqrt{(-9900)^2 + 10100^2} \angle \tan^{-1}(10100/-9900)}$$

$$G(j\omega) = \frac{\sqrt{10100} \angle 84,289^\circ}{\sqrt{200020000} \angle 134,43^\circ}$$

$$G(j\omega) = \frac{100,5 \angle 84,289^\circ}{14143 \angle 134,43^\circ} = 0,007106 \angle -50,138^\circ$$

$$|G(j\omega)|_{dB} = 20 \log(0,007106) = -42,968 \text{ dB}$$

$$\angle G(j\omega) = -50,138^\circ$$

Exemplo 2) $G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$

1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} \sigma(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Valores em $\omega = 1000 \text{ rad/s}$

(1 década acima do pólo em $s = -100$):

$$G(s) = \frac{s + 10}{(s + 1)(s + 100)} = \frac{s + 10}{s^2 + 101s + 100}$$

$$G(j\omega) = \frac{10 + j1000}{(j1000)^2 + j10100 + 100} = \frac{10 + j1000}{100 - 1000000 + j10100}$$

$$G(j\omega) = \frac{10 + j1000}{-999900 + j10100}$$

$$G(j\omega) = \frac{\sqrt{1000100} \angle 89,427^\circ}{\sqrt{9,999 \times 10^{11}} \angle 179,42^\circ}$$

$$G(j\omega) = \frac{1000 \angle 89,427^\circ}{999950 \angle 179,42^\circ} = \underbrace{0.0010001}_{\cong 1/1000} \angle -89,994^\circ$$

e

$$|G(j\omega)|_{\text{dB}} = 20 \log(0.0010001) = \underbrace{-59,999}_{\cong -60} \text{ dB}$$

$$\angle G(j\omega) \cong -90^\circ$$

$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s + 10)}{(s + 1)(s + 100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

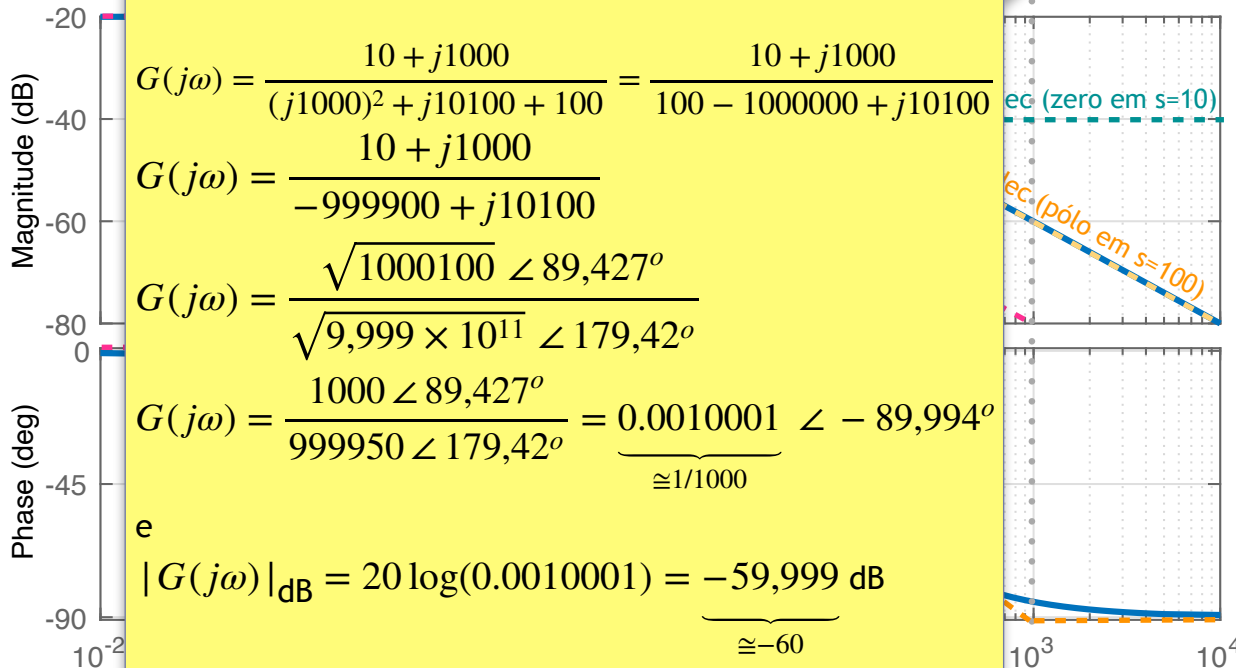
Degrau

$$\text{DC} \left[\frac{\text{dB}}{\text{dB}} \right] = 20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$$

rar que:

pólo decreta ganho de: -20 dB/déc ;

zero incrementa ganho de: $+20 \text{ dB/déc}$.



```
>> G=tf([1 10],poly([-1 -100]))
G =
```

$$\frac{s + 10}{s^2 + 101s + 100}$$

```
>> zpk(G)
```

$$\frac{(s+10)}{(s+100)(s+1)}$$

```
>> bode(G)
```

```
>> grid
```

Exemplo 2) $G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$

1. Cálculo do ganho “DC” (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Neste caso: Ganho DC quando aplicado degrau

$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s+10)}{(s+1)(s+100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

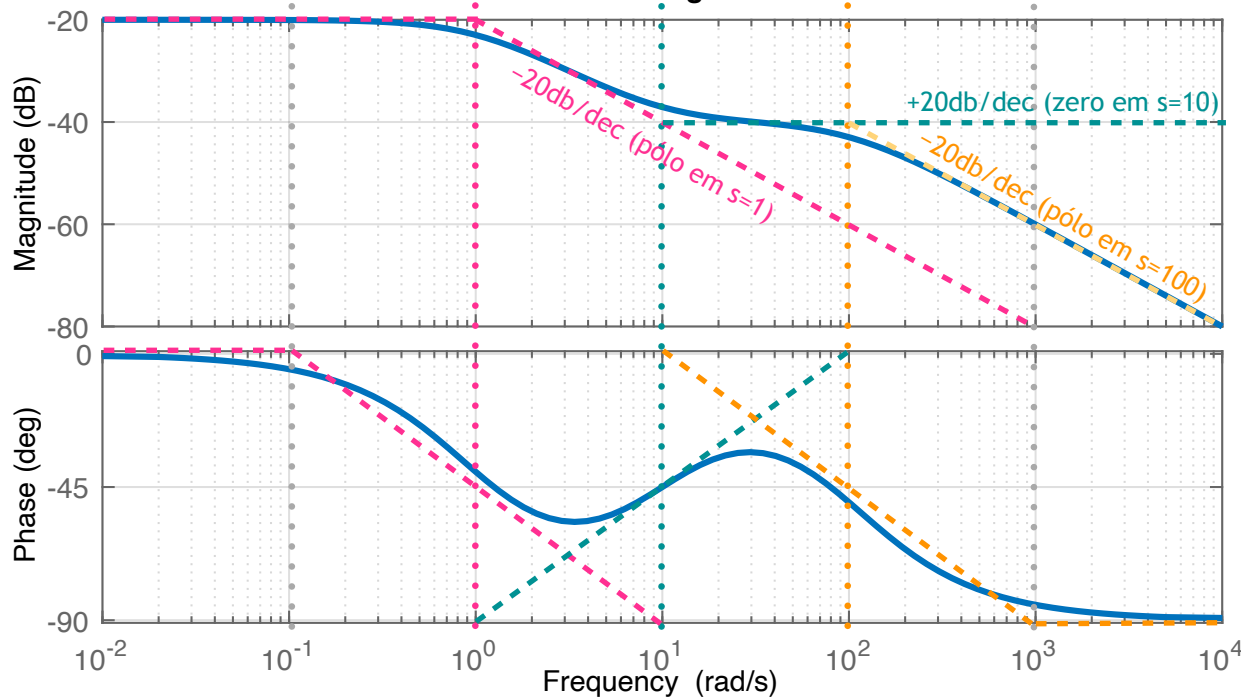
Ganho DC $\left|_{\text{dB}}^{\text{Degrau}} = 20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$

2. Lembrar que:

cada pólo decreta ganho de: -20 db/déc;

cada zero incrementa ganho de: +20 db/déc.

Bode Diagram



```
>> G=tf([1 10],poly([-1 -100]))
G =
```

$$\frac{s + 10}{s^2 + 101s + 100}$$

```
>> zpk(G)
```

$$\frac{(s+10)}{(s+100)(s+1)}$$

```
>> bode(G)
```

```
>> grid
```

Outros exemplos:

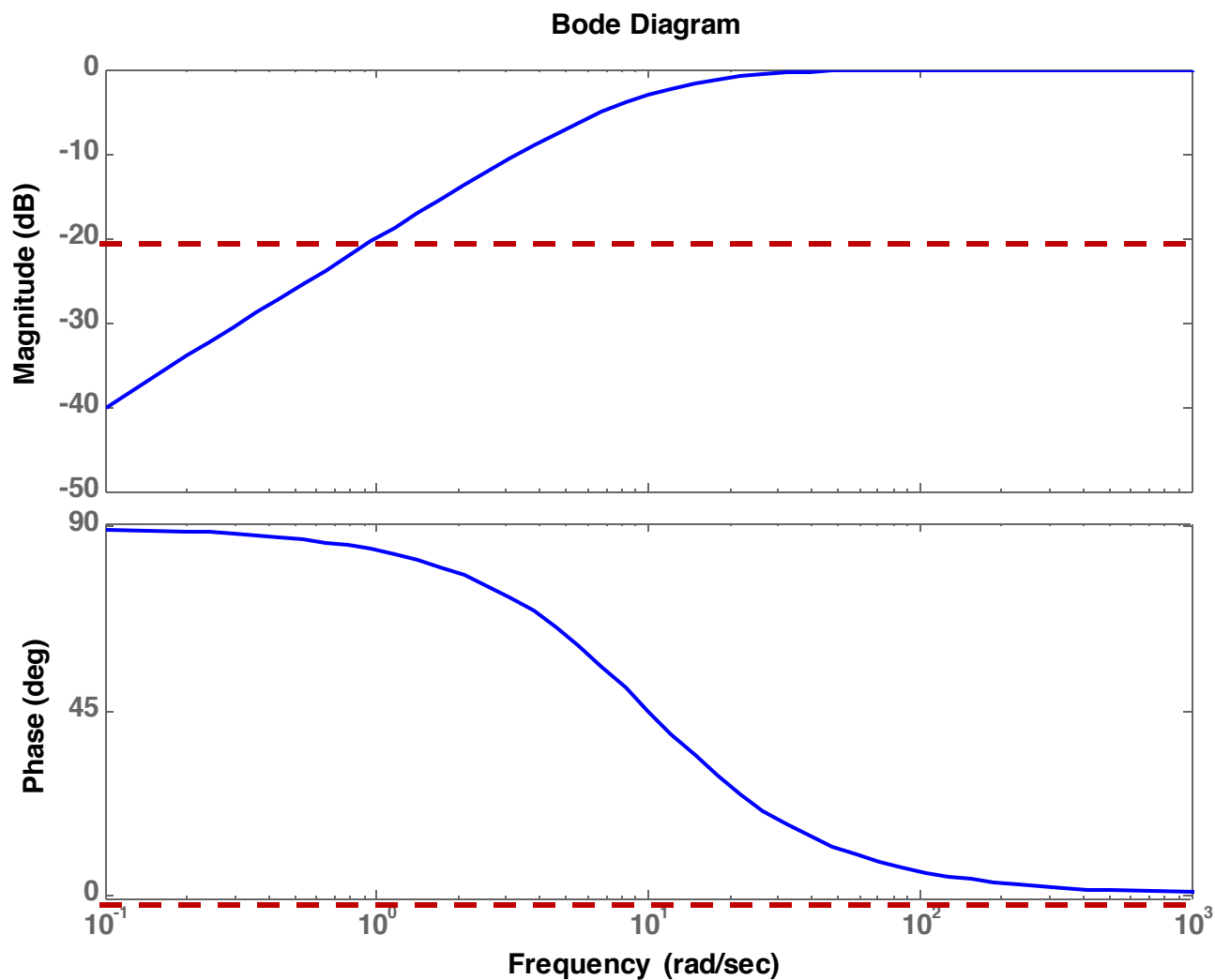
$$\triangleright G_1(s) = \frac{s}{(s+10)}$$

$$G_1(s) = \frac{1}{10} \cdot \frac{s}{\left(\frac{s}{10} + 1\right)}$$

Linha de base,
ganho = 0,1
(-20 dB)

$$\log(10) = 1$$

$$\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$$



Outros exemplos:

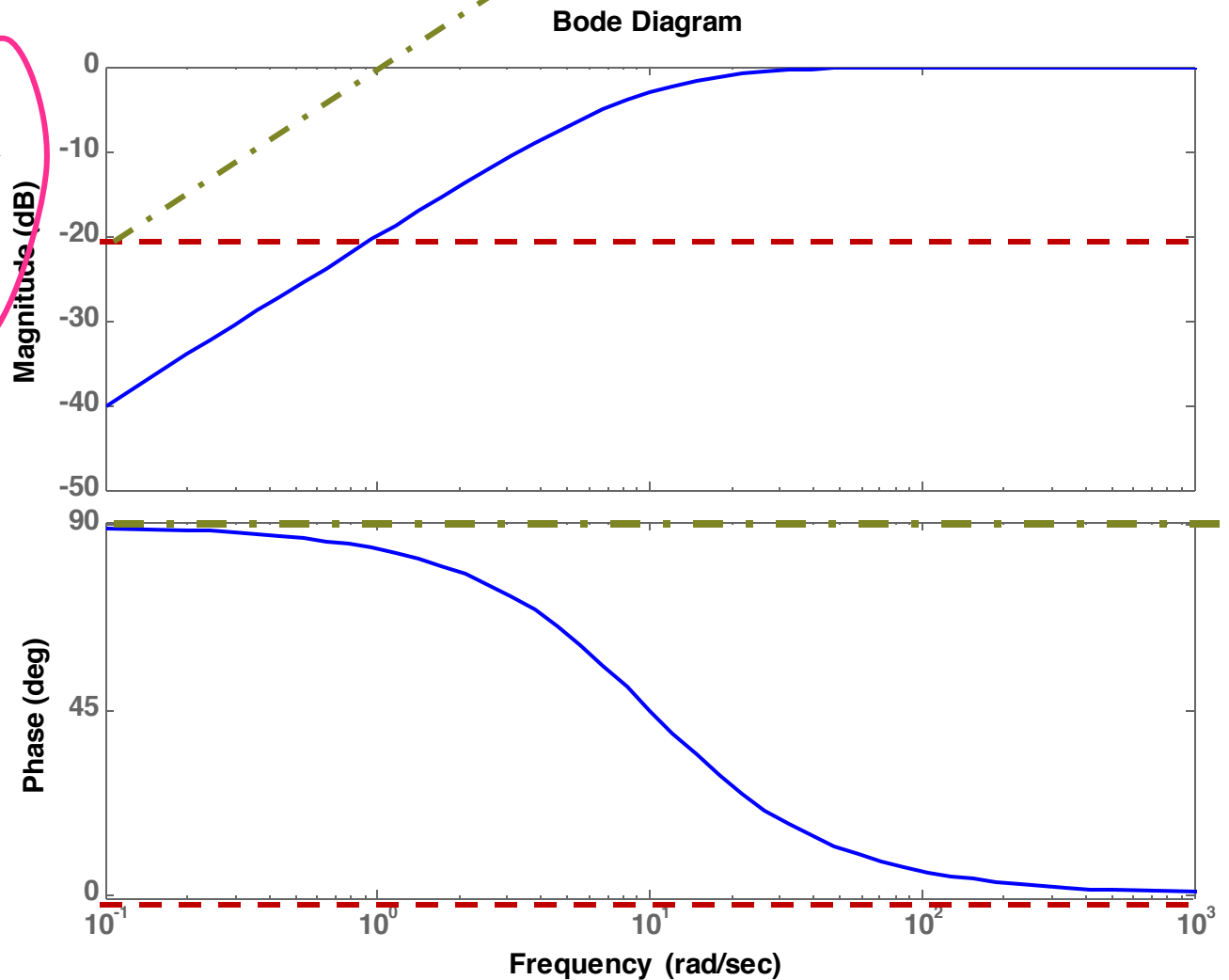
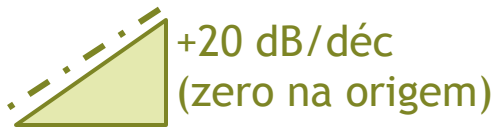
► $G_1(s) = \frac{s}{(s+10)}$

$$G_1(s) = \frac{1}{10} \cdot \frac{s}{\left(\frac{s}{10} + 1\right)}$$

Zero na origem

$$\log(10) = 1$$

$$\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$$



Outros exemplos:

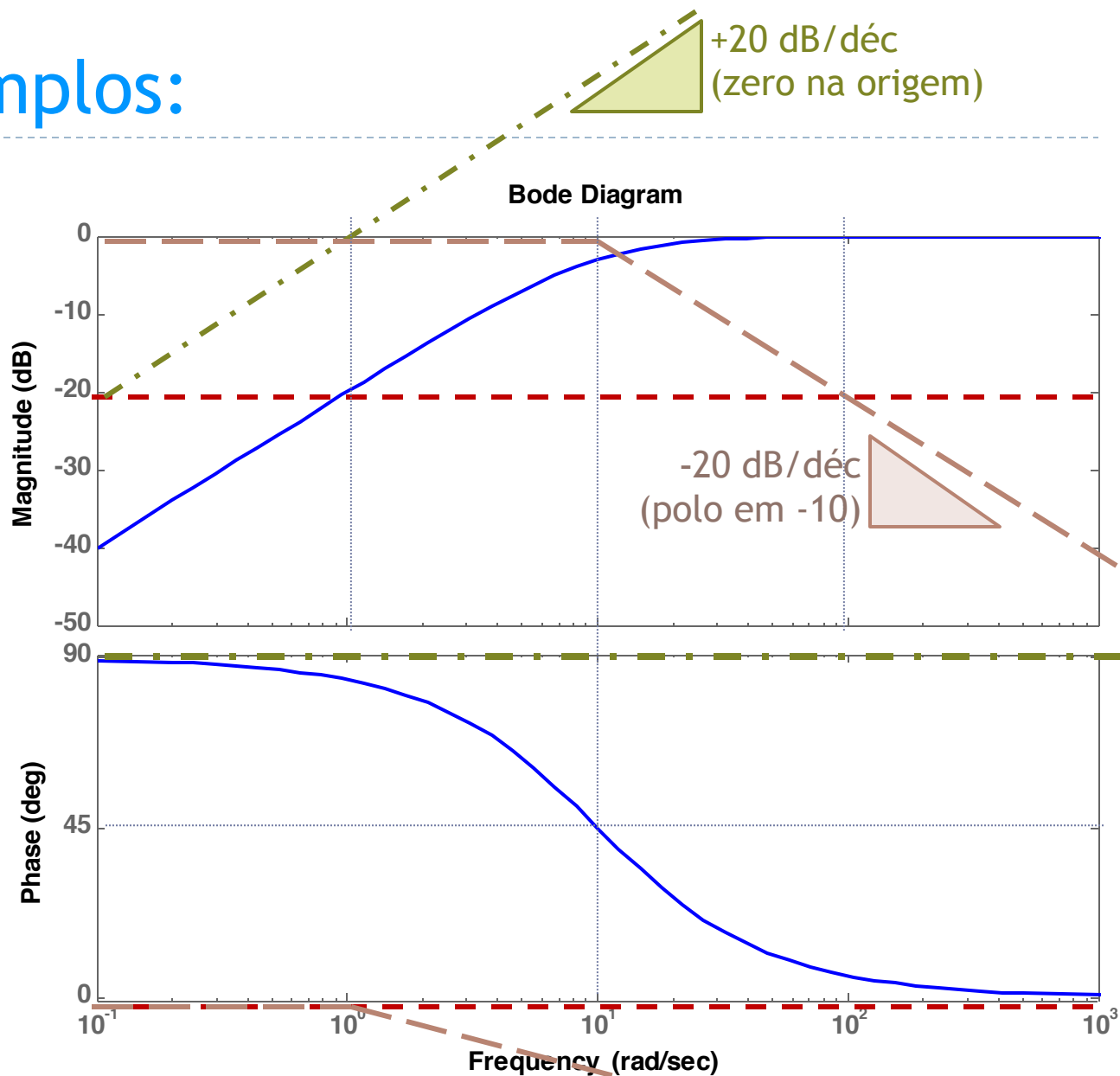
$$G_1(s) = \frac{s}{(s+10)}$$

$$G_1(s) = \frac{1}{10} \cdot \frac{s}{\left(\frac{s}{10} + 1\right)}$$

Polo real em $s = -10$

$$\log(10) = 1$$

$$\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$$



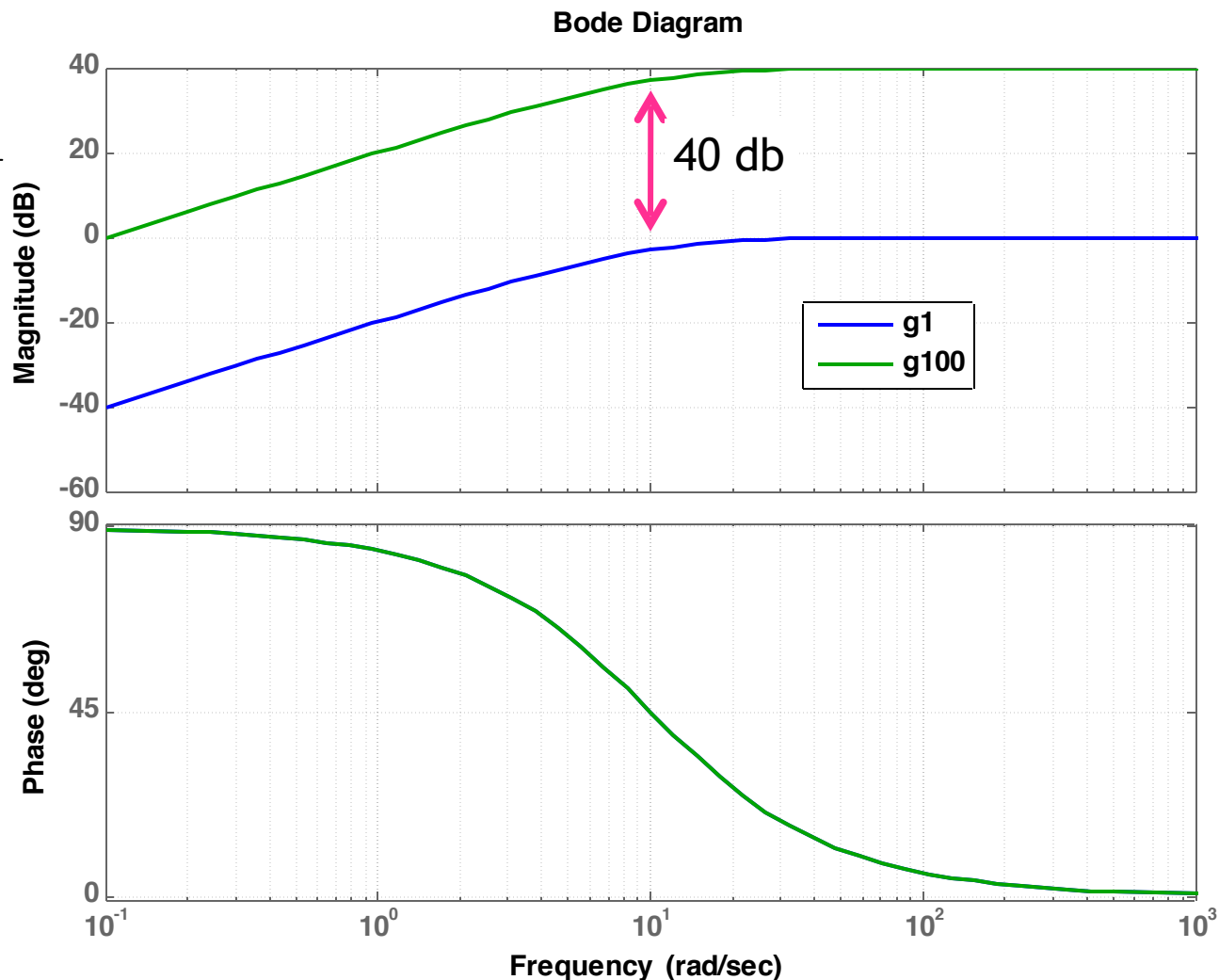
Outros exemplos:

► $G_1(s) = \frac{s}{(s+10)}$

$$G_{100}(s) = 100 \frac{s}{(s+10)}$$

$$G_{100}(s) = 10 \frac{s}{s + \frac{s}{10}}$$

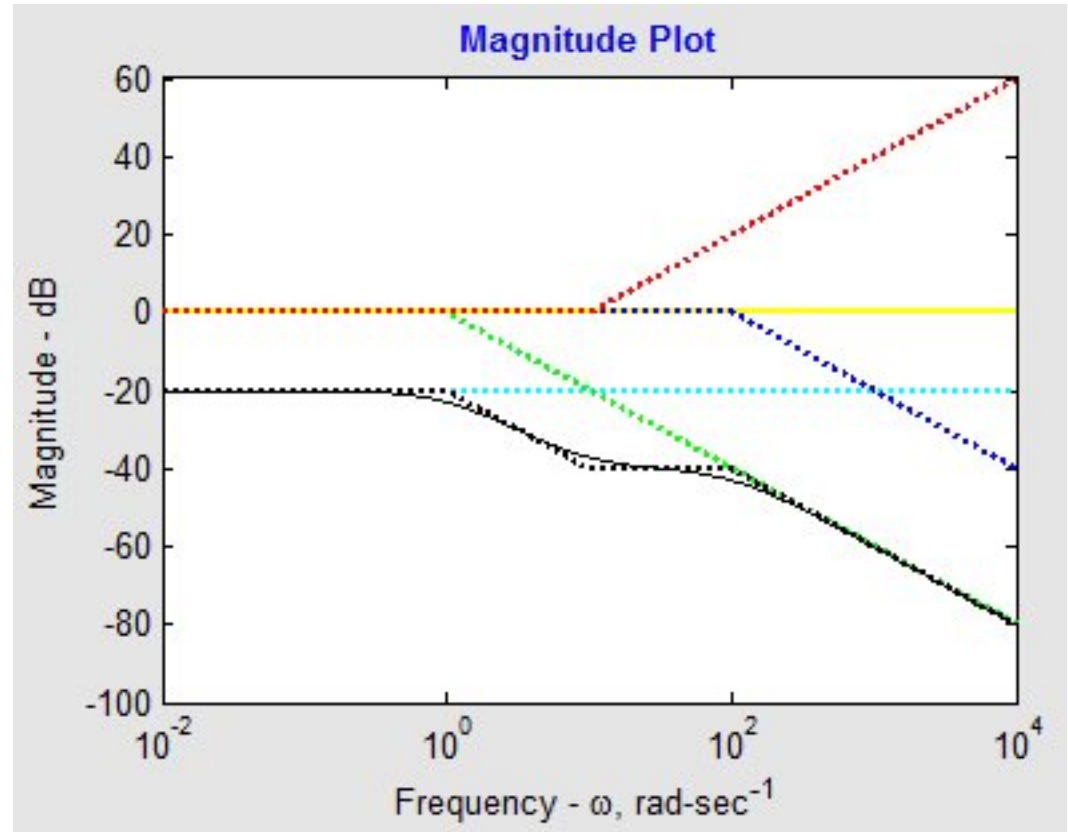
Diferença:
Ganho de magnitude:
 $\log(100) = 2$



Outros exemplos:

$$\begin{aligned} \blacktriangleright G_2(s) &= \frac{s+10}{(s+1)(s+100)} \\ &= \frac{10\left(\frac{s}{10}+1\right)}{(s+1)\cdot 100\left(\frac{s}{100}+1\right)} \\ &= \frac{10}{100} \cdot \frac{\left(\frac{s}{10}+1\right)}{(s+1)\left(\frac{s}{100}+1\right)} \end{aligned}$$

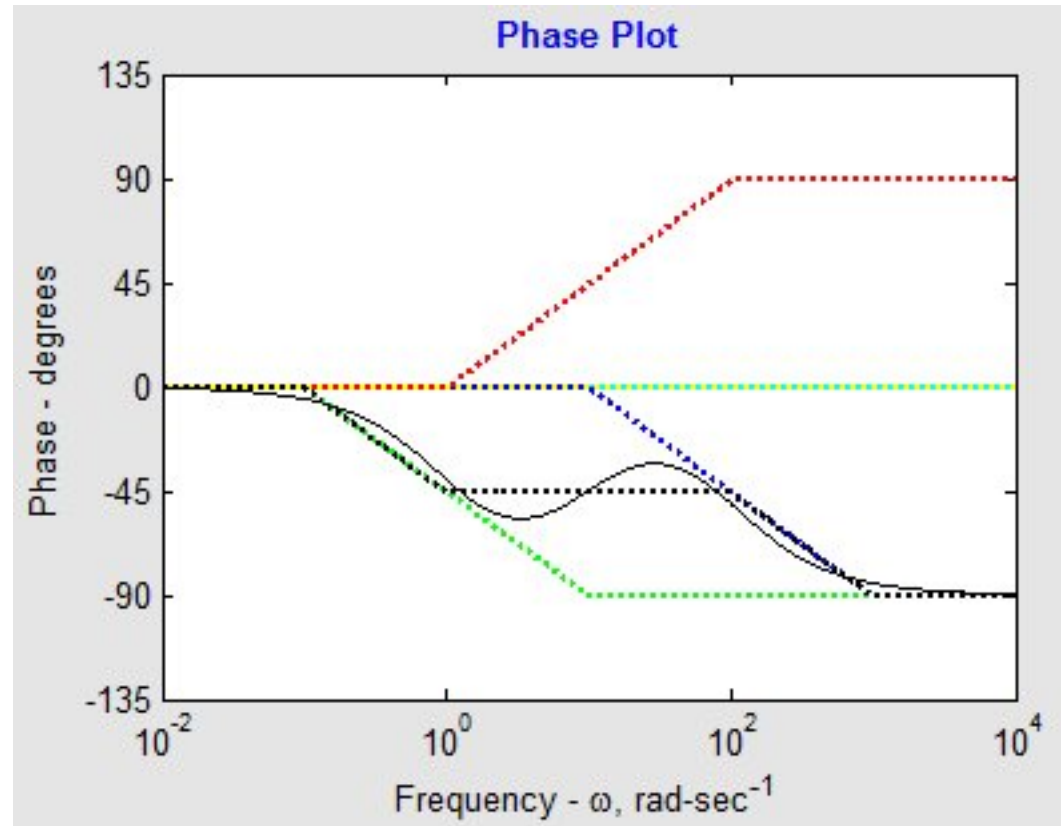
- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 0.1 (-20 dB)
- Real Pole at -1e+002
- Real Pole at -1
- Real Zero at -10



Outros exemplos:

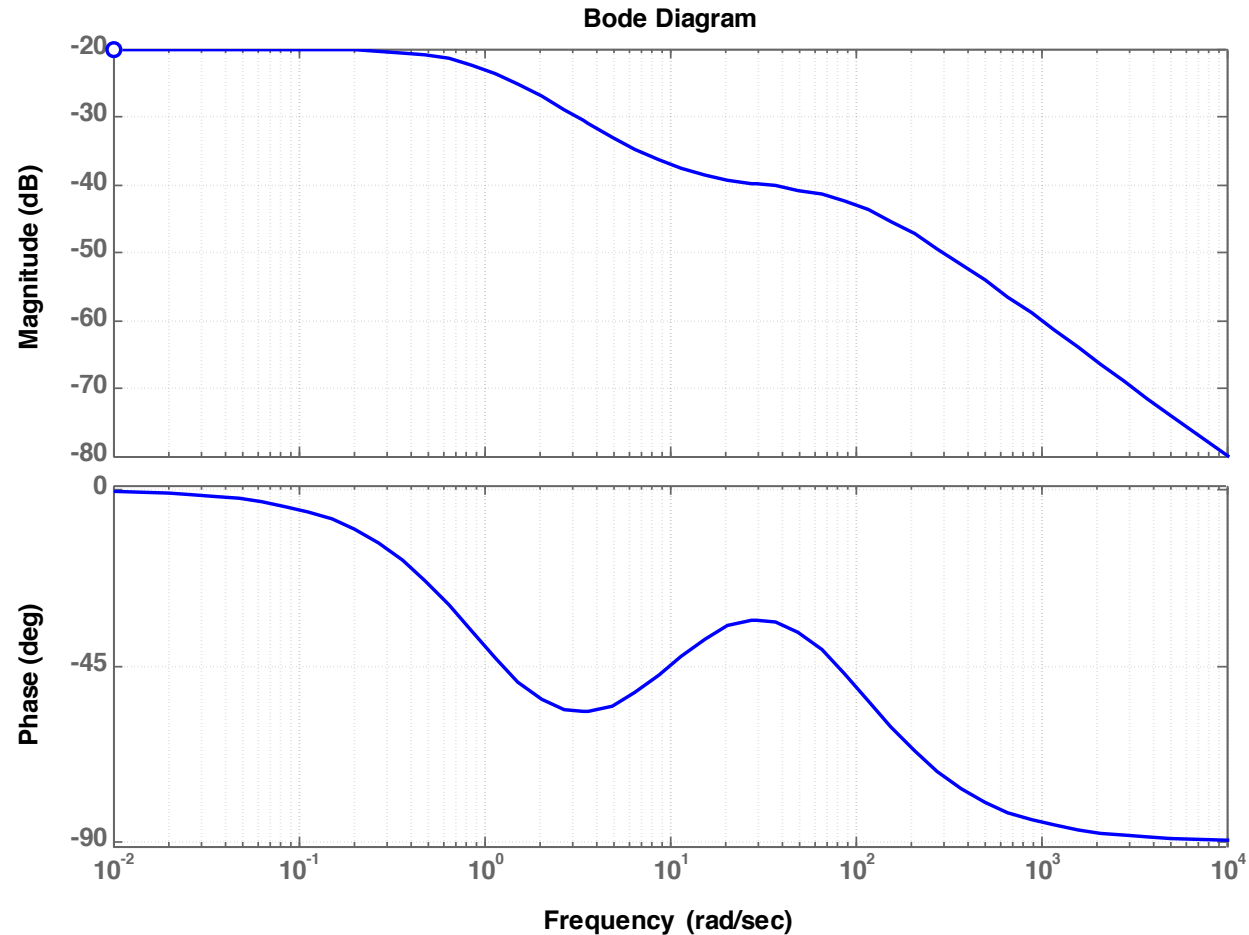
$$\begin{aligned}
 \blacktriangleright G_2(s) &= \frac{s+10}{(s+1)(s+100)} \\
 &= \frac{10\left(\frac{s}{10}+1\right)}{(s+1)\cdot 100\left(\frac{s}{100}+1\right)} \\
 &= \frac{10}{100} \cdot \frac{\left(\frac{s}{10}+1\right)}{(s+1)\left(\frac{s}{100}+1\right)}
 \end{aligned}$$

- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 0.1 (-20 dB)
- Real Pole at -1e+002
- Real Pole at -1
- Real Zero at -10



Outros exemplos:

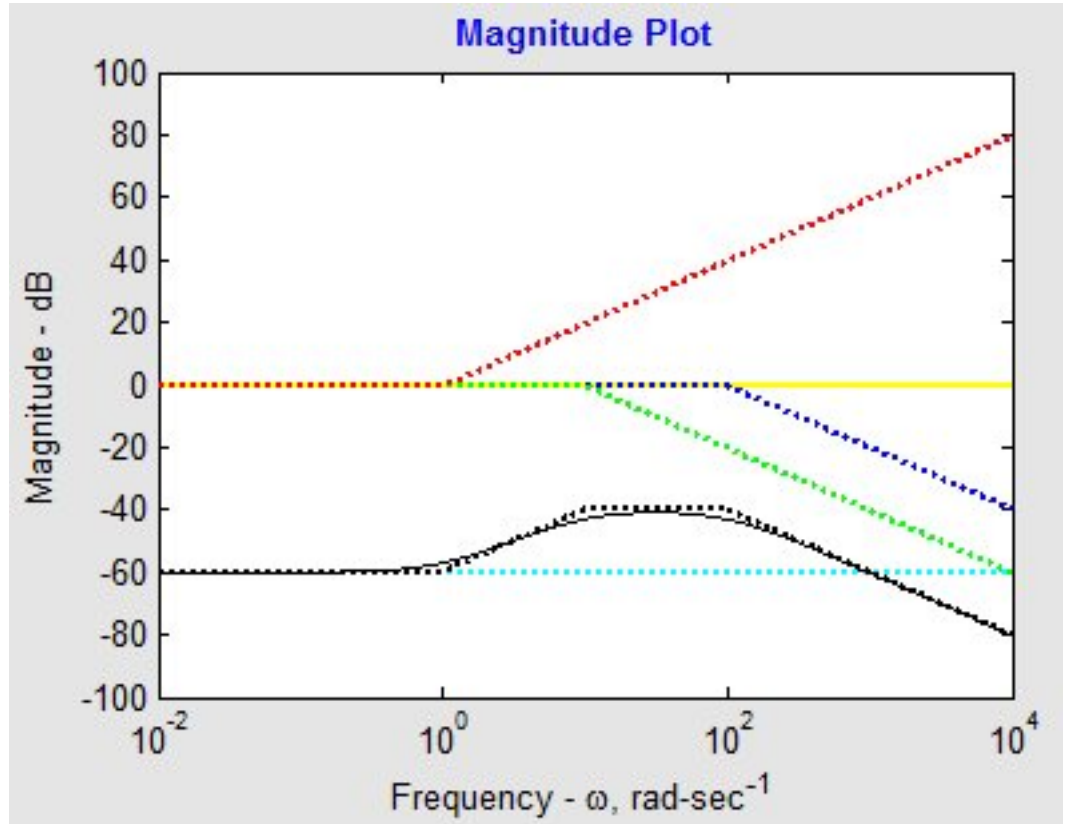
$$\begin{aligned} \blacktriangleright G_2(s) &= \frac{s + 10}{(s + 1)(s + 100)} \\ &= \frac{10 \left(\frac{s}{10} + 1 \right)}{(s + 1) \cdot 100 \left(\frac{s}{100} + 1 \right)} \\ &= \frac{10}{100} \cdot \frac{\left(\frac{s}{10} + 1 \right)}{(s + 1) \left(\frac{s}{100} + 1 \right)} \end{aligned}$$



Outros Exemplos:

$$\begin{aligned}
 \triangleright G_3(s) &= \frac{s+1}{(s+10)(s+100)} \\
 &= \frac{(s+1)}{(s+1)} \\
 &= \frac{10\left(\frac{s}{10}+1\right) \cdot 100\left(\frac{s}{100}+1\right)}{(s+1)} \\
 &= \frac{1000\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}{(s+1)}
 \end{aligned}$$

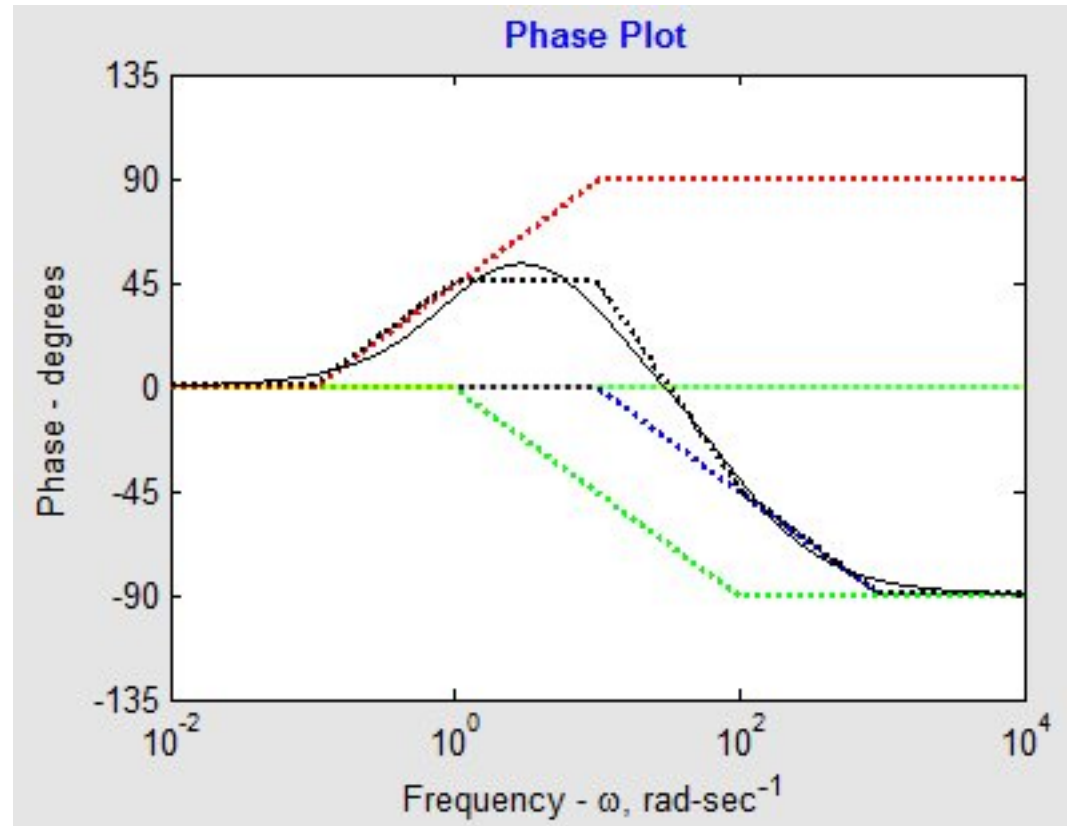
- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 0.001 (-60 dB)
- Real Pole at -1e+002
- Real Pole at -10
- Real Zero at -1



Outros Exemplos:

$$\begin{aligned}
 \triangleright G_3(s) &= \frac{s+1}{(s+10)(s+100)} \\
 &= \frac{(s+1)}{10\left(\frac{s}{10}+1\right) \cdot 100\left(\frac{s}{100}+1\right)} \\
 &= \frac{1000\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}{(s+1)}
 \end{aligned}$$

- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 0.001 (-60 dB)
- Real Pole at -1e+002
- Real Pole at -10
- Real Zero at -1



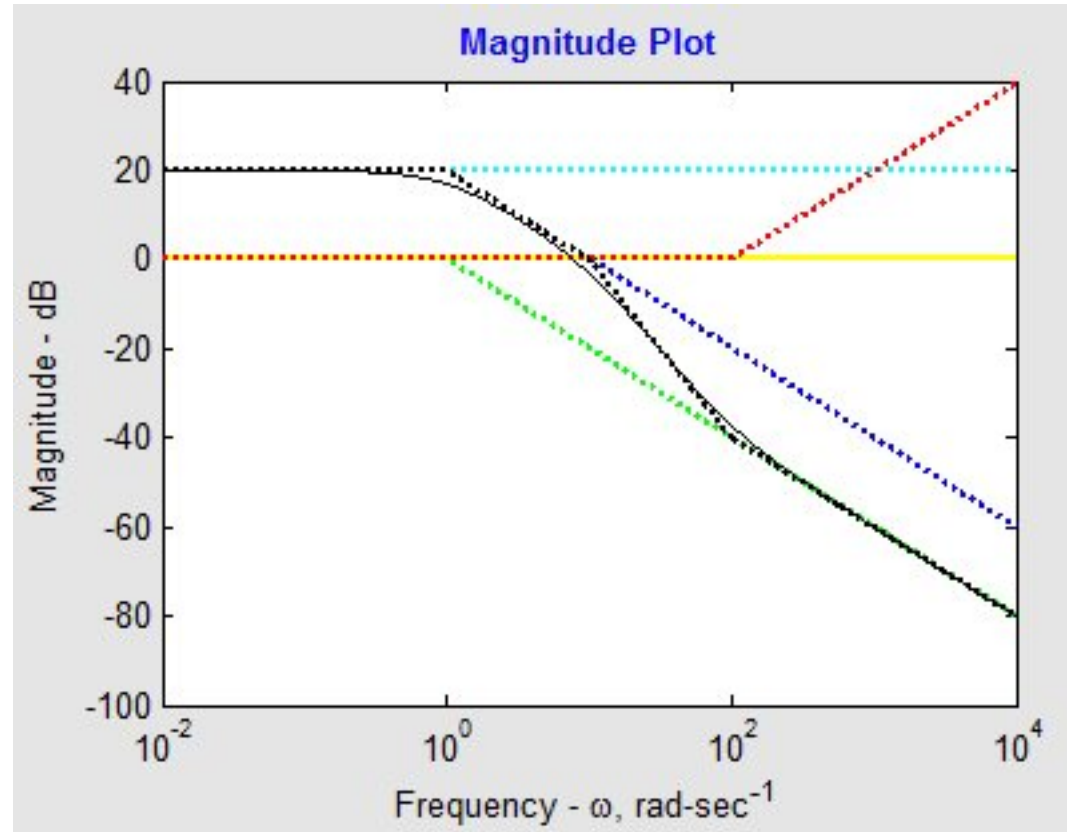
Outros Exemplos:

$$\triangleright G_4(s) = \frac{s + 100}{(s + 1)(s + 10)}$$

$$= \frac{100 \left(\frac{s}{100} + 1 \right)}{(s + 1) \cdot 10 \left(\frac{s}{10} + 1 \right)}$$

$$= 10 \frac{\left(\frac{s}{100} + 1 \right)}{(s + 1) \cdot \left(\frac{s}{10} + 1 \right)}$$

- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 10 (20 dB)
- Real Pole at -10
- Real Pole at -1
- Real Zero at -1e+002



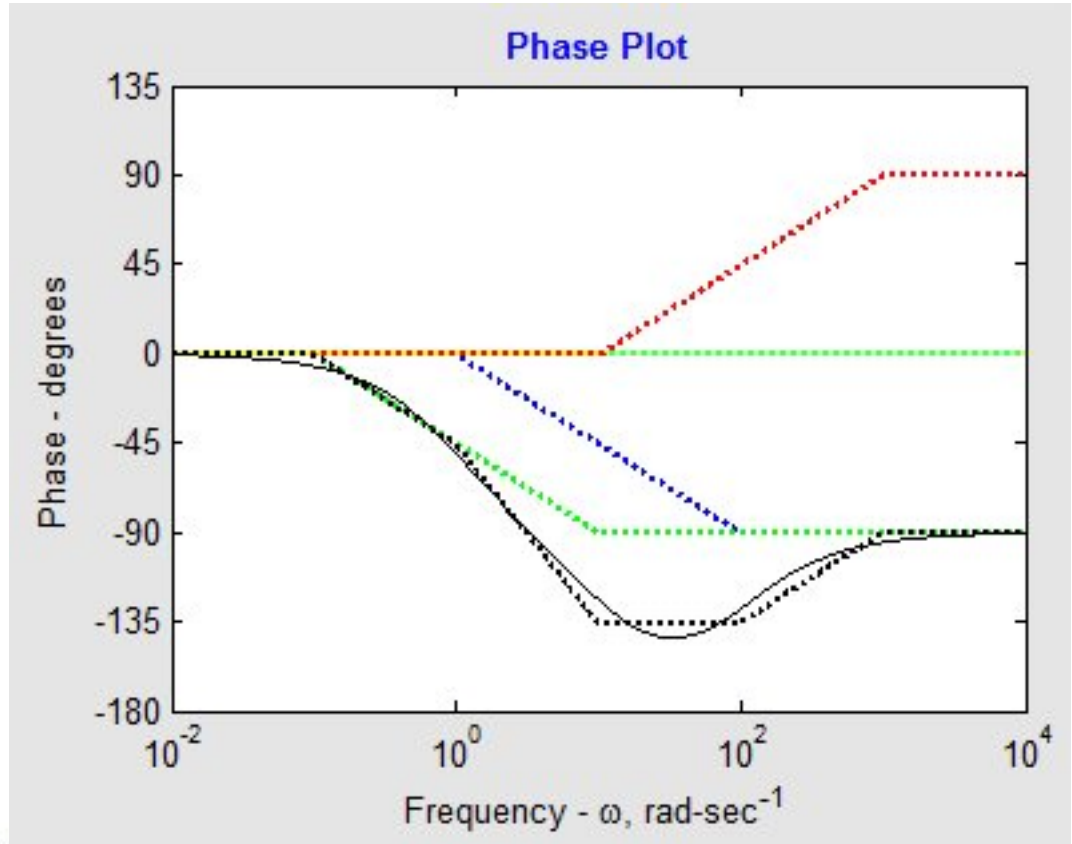
Outros Exemplos:

$$\triangleright G_4(s) = \frac{s + 100}{(s + 1)(s + 10)}$$

$$= \frac{100 \left(\frac{s}{100} + 1 \right)}{(s + 1) \cdot 10 \left(\frac{s}{10} + 1 \right)}$$

$$= 10 \frac{\left(\frac{s}{100} + 1 \right)}{(s + 1) \cdot \left(\frac{s}{10} + 1 \right)}$$

- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 10 (20 dB)
- Real Pole at -10
- Real Pole at -1
- Real Zero at -1e+002



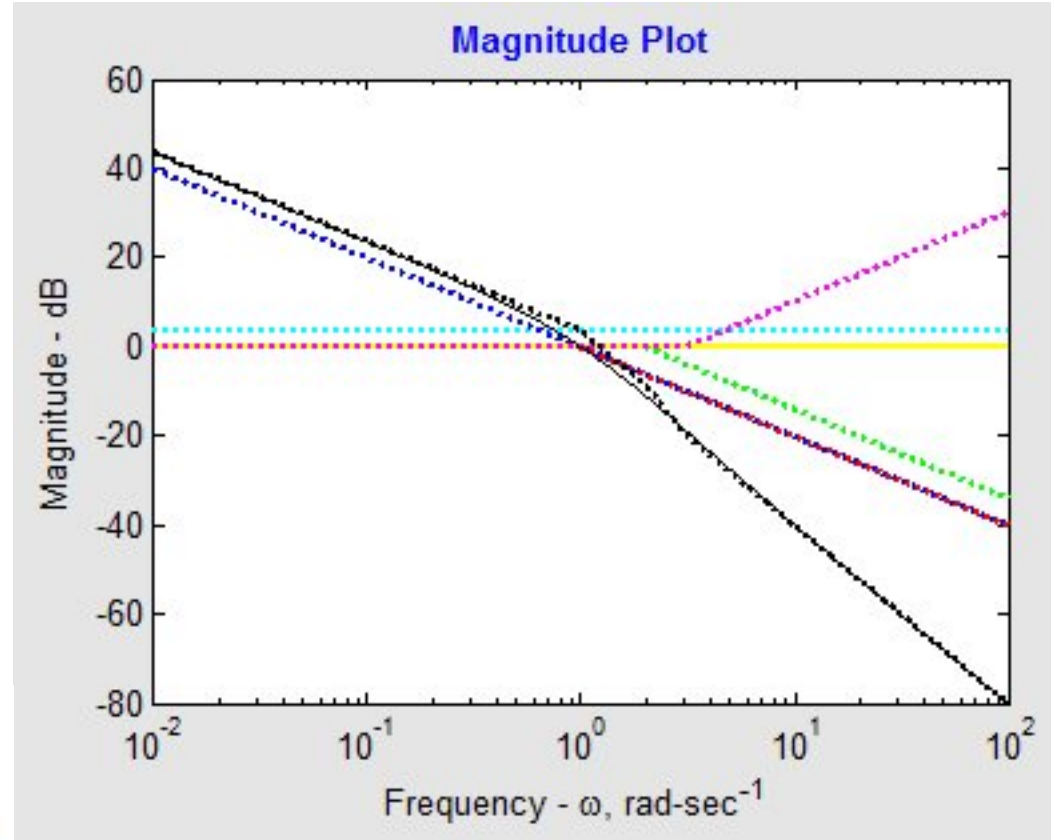
Outros Exemplos:

$$\triangleright G_5(s) = \frac{s+3}{s(s+1)(s+2)}$$

$$= \frac{3\left(\frac{s}{3}+1\right)}{s(s+1) \cdot 2\left(\frac{s}{2}+1\right)}$$

$$= \frac{3}{2} \cdot \frac{\left(\frac{s}{3}+1\right)}{s(s+1) \cdot \left(\frac{s}{2}+1\right)}$$

- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 1.5 (3.5 dB)
- Pole at origin
- Real Pole at -2
- Real Pole at -1
- Real Zero at -3



$$20 \times \log(3/2) = 20 \times 0,1761 = 3,5218$$

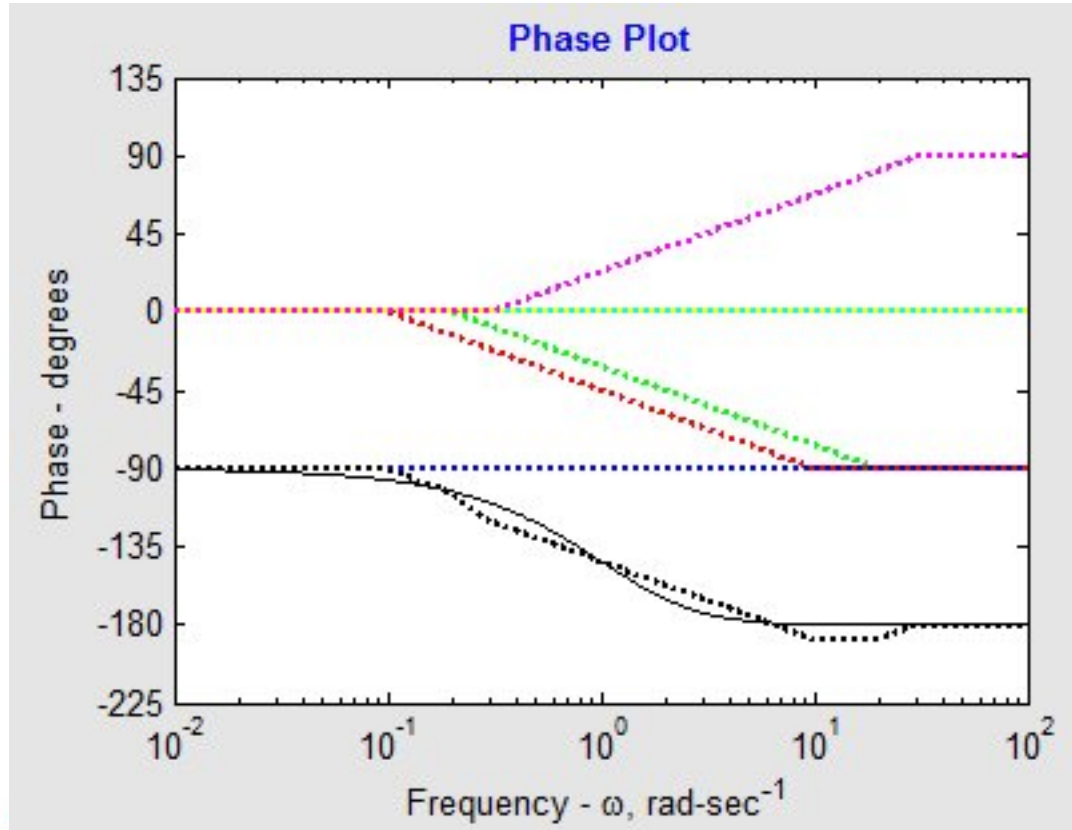
Outros Exemplos:

$$\triangleright G_5(s) = \frac{s+3}{s(s+1)(s+2)}$$

$$= \frac{3\left(\frac{s}{3}+1\right)}{s(s+1)\cdot 2\left(\frac{s}{2}+1\right)}$$

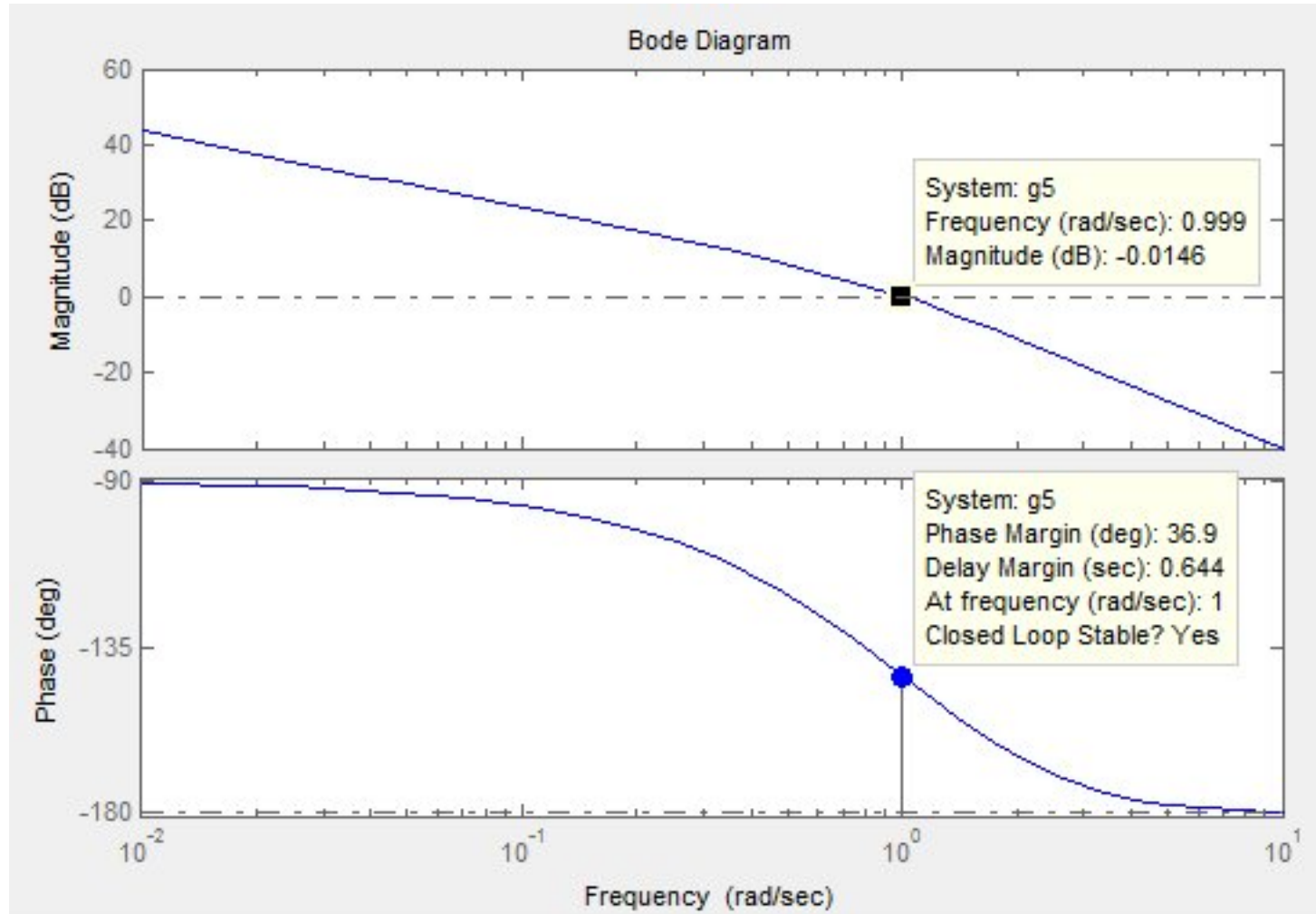
$$= \frac{3}{2} \cdot \frac{\left(\frac{s}{3}+1\right)}{s(s+1)\cdot\left(\frac{s}{2}+1\right)}$$

- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 1.5 (3.5 dB)
- Pole at origin
- Real Pole at -2
- Real Pole at -1
- Real Zero at -3



Outros Exemplos:

$$\begin{aligned} G_5(s) &= \frac{s+3}{s(s+1)(s+2)} \\ &= \frac{3\left(\frac{s}{3}+1\right)}{s(s+1)\cdot 2\left(\frac{s}{2}+1\right)} \\ &= \frac{3}{2} \cdot \frac{\left(\frac{s}{3}+1\right)}{s(s+1)\cdot\left(\frac{s}{2}+1\right)} \end{aligned}$$



Sistemas com polos complexos

▶ Seja:

$$G(s) = s^2 + 2\zeta w_n s + w_n^2 = w_n^2 \left(\frac{s^2}{w_n^2} + 2\zeta \frac{s}{w_n} + 1 \right)$$

▶ Em baixas frequências:

$$G(s) \approx w_n^2 = w_n^2 \angle 0^\circ \qquad 20 \log M = 20 \log |G(jw)| = 20 \log w_n^2$$

▶ Em altas frequências:

$$G(s) \approx s^2 \qquad 20 \log M = 20 \log |G(jw)| = 20 \log w^2 = 40 \log w$$
$$G(jw) \approx -w^2 = w^2 \angle 180^\circ$$

▶ Detalhes:

▶ w_n : frequência de corte (quebra).

▶ Fase em w_n :

$$G(jw) = s^2 + 2\zeta w_n s + w_n^2 \Big|_{s \rightarrow jw} = (w_n^2 - w^2) + j2\zeta w_n w$$

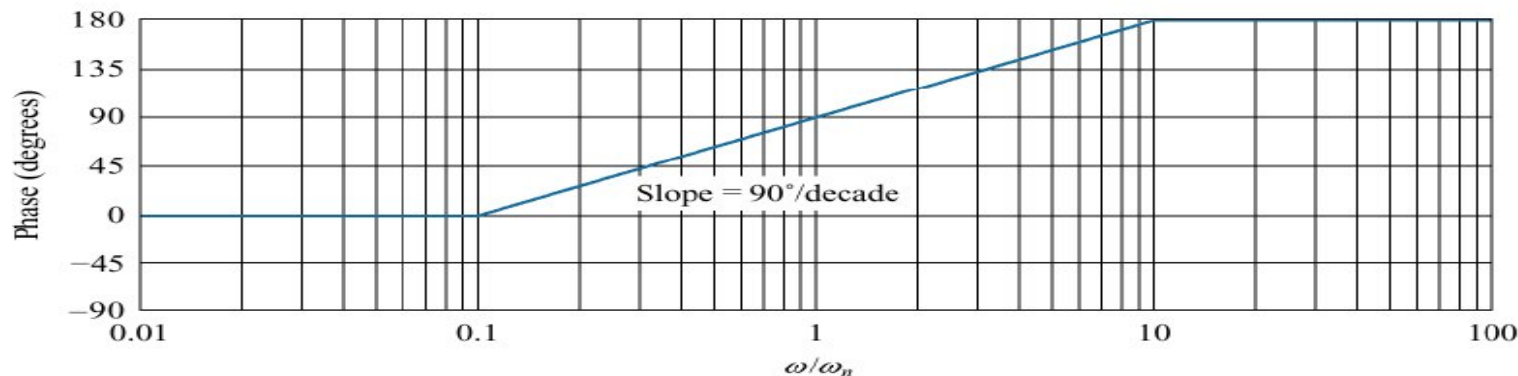
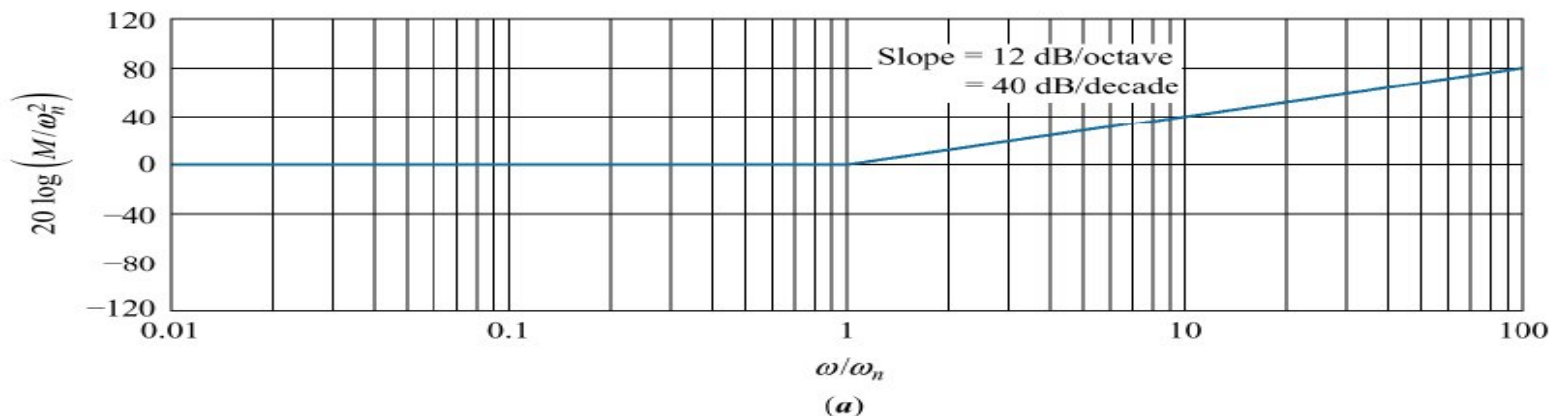
em w_n o resultado é: $j2\zeta w_n^2$ assim a fase na frequência natural é de $+90^\circ$

Sistemas com polos complexos

▶ Seja:
$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = \omega_n^2 \left(\frac{s^2}{\omega_n^2} + 2\zeta \frac{s}{\omega_n} + 1 \right)$$

▶ Em baixas frequências: $G(s) \approx \omega_n^2 = \omega_n^2 \angle 0^\circ$ $20 \log M = 20 \log |G(j\omega)| = 20 \log \omega_n^2$

▶ Em altas frequências: $G(s) \approx s^2$ $20 \log M = 20 \log |G(j\omega)| = 20 \log \omega^2 = 40 \log \omega$
 $G(j\omega) \approx -\omega^2 = \omega^2 \angle 180^\circ$



Sistemas com polos complexos

Correções em diagrama assintótico..

▶ Seja:

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = \omega_n^2 \left(\frac{s^2}{\omega_n^2} + 2\zeta \frac{s}{\omega_n} + 1 \right)$$

▶ Um polinômio de 1ª-ordem resulta numa diferença não superior a **3,01 dB** na magnitude e **5,71°** em relação à fase (no ponto do polo).

▶ Um polinômio de 2ª-ordem pode implicar maior disparidade, depende do valor de ζ (na localização dos polos complexos):

$$G(j\omega) = s^2 + 2\zeta\omega_n s + \omega_n^2 \Big|_{s \rightarrow j\omega} = (\omega_n^2 - \omega^2) + j2\zeta\omega_n \omega$$

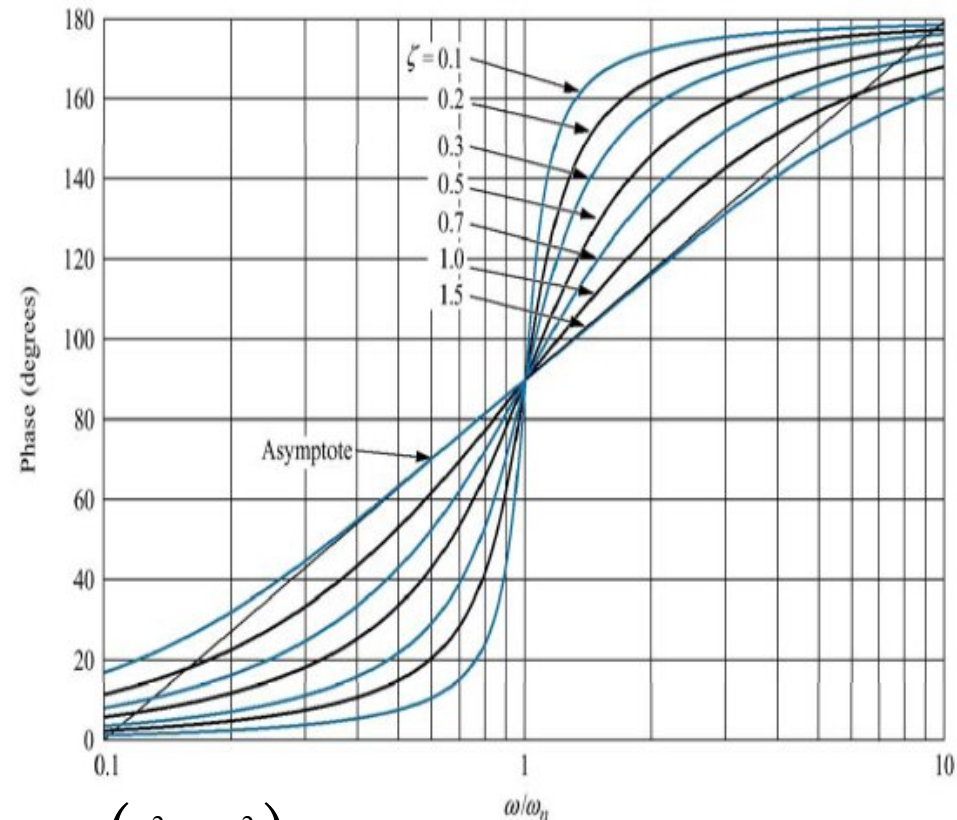
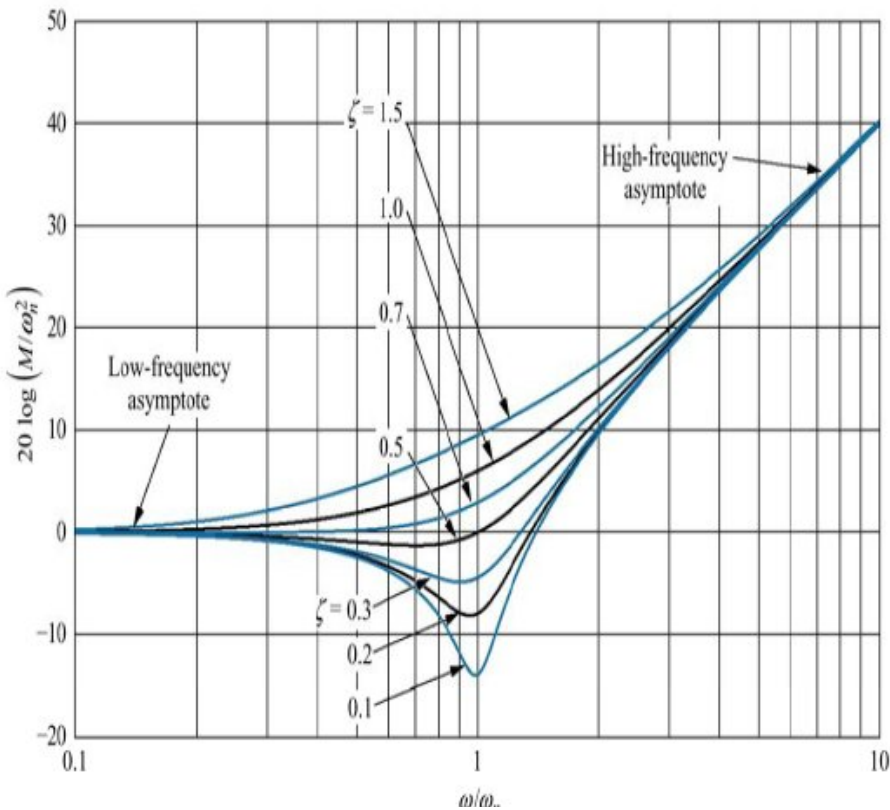
$$M = \sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n \omega)^2}$$

$$Fase = \tan^{-1} \left(\frac{2\zeta\omega_n \omega}{\omega_n^2 - \omega^2} \right)$$

Sistemas com polos complexos

Correções em diagrama assintótico..

- ▶ Seja: $G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = \omega_n^2 \left(\frac{s^2}{\omega_n^2} + 2\zeta \frac{s}{\omega_n} + 1 \right)$
- $$M = \sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$
- $$Fase = \tan^{-1} \left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \right)$$



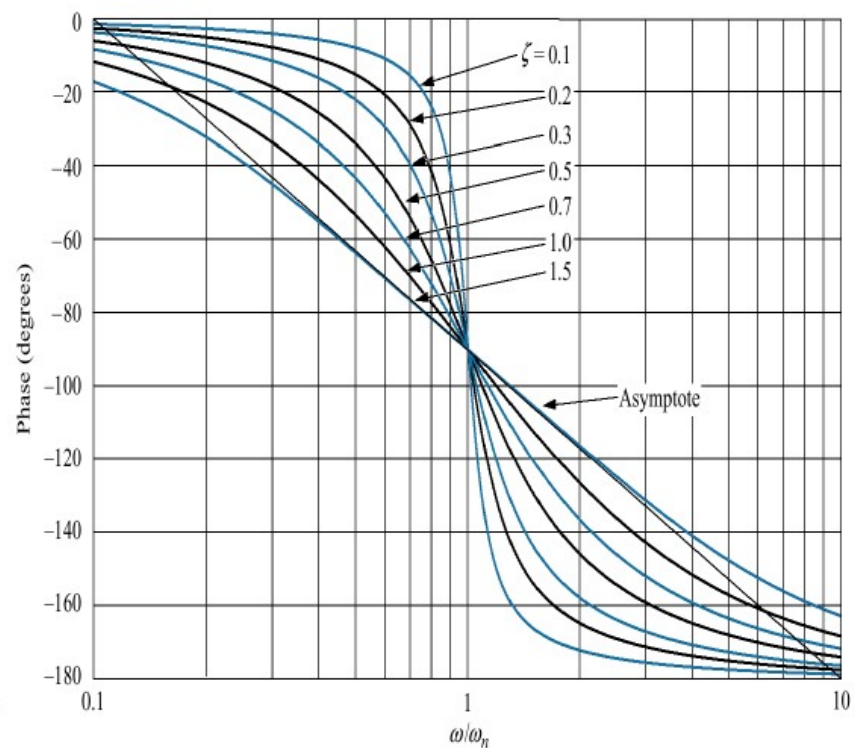
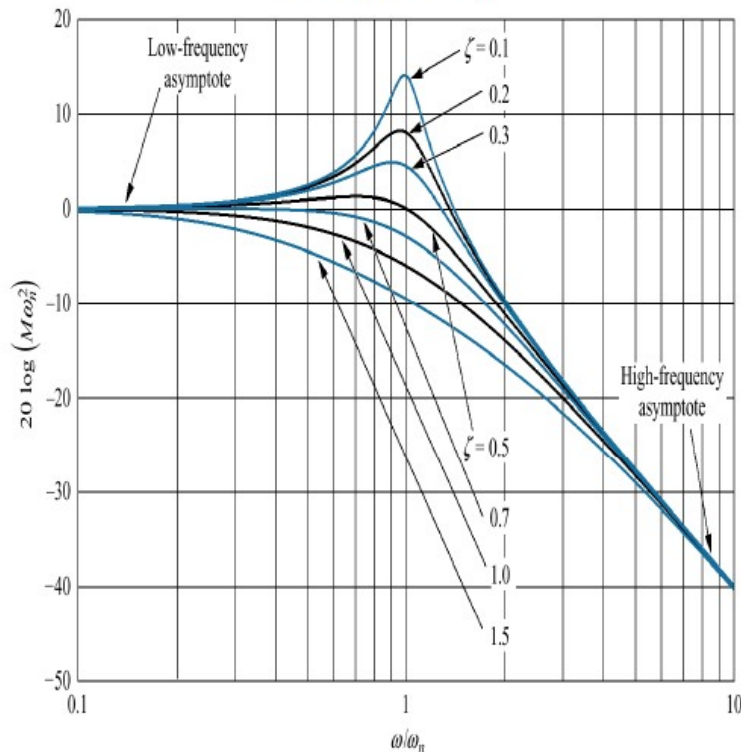
$$G(j\omega) = s^2 + 2\zeta\omega_n s + \omega_n^2 \Big|_{s \rightarrow j\omega} = (\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega$$

Sistemas com polos complexos

Correções em diagrama assintótico..

$$G(s) = 1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$$

- The slope is -40dB/decade .
- The normalized magnitude at the scaled natural frequency is $-20\log 2\zeta\omega_n^2$



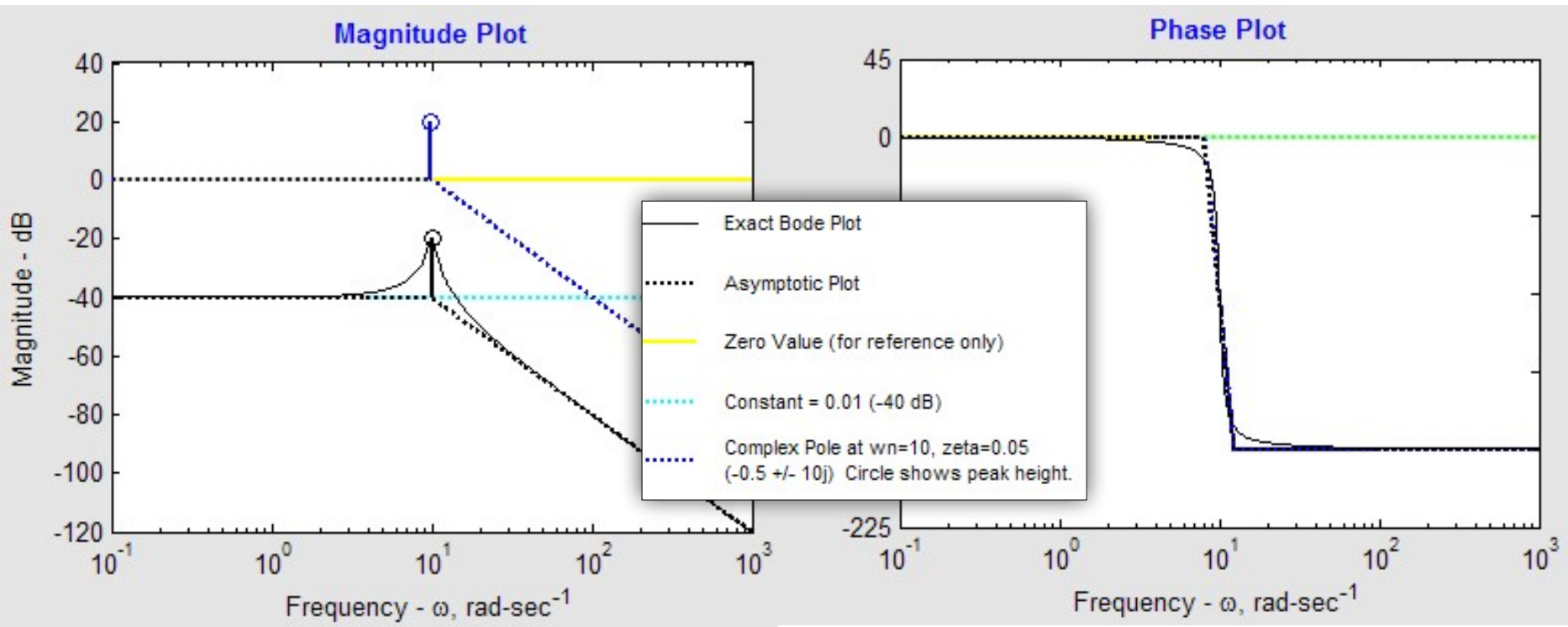
Exemplo polos complexos

$$G(s) = \frac{1}{s^2 + s + 100}$$

$$G(s) = \frac{1}{(s + 0,5 + j10)(s + 0,5 - j10)}$$

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = \omega_n^2 \left(\frac{s^2}{\omega_n^2} + 2\zeta \frac{s}{\omega_n} + 1 \right)$$

$$\omega_n = \sqrt{100} = 10 \quad \zeta = 1/2\omega_n = 1/20 = 0,05$$



$$20 \cdot \log_{10}(1/100) = -40 \text{ dB}$$

Problemas sugeridos:

```
>> roots([1 3 50])
ans =
    -1.5000 + 6.9101i
    -1.5000 - 6.9101i
>>
```

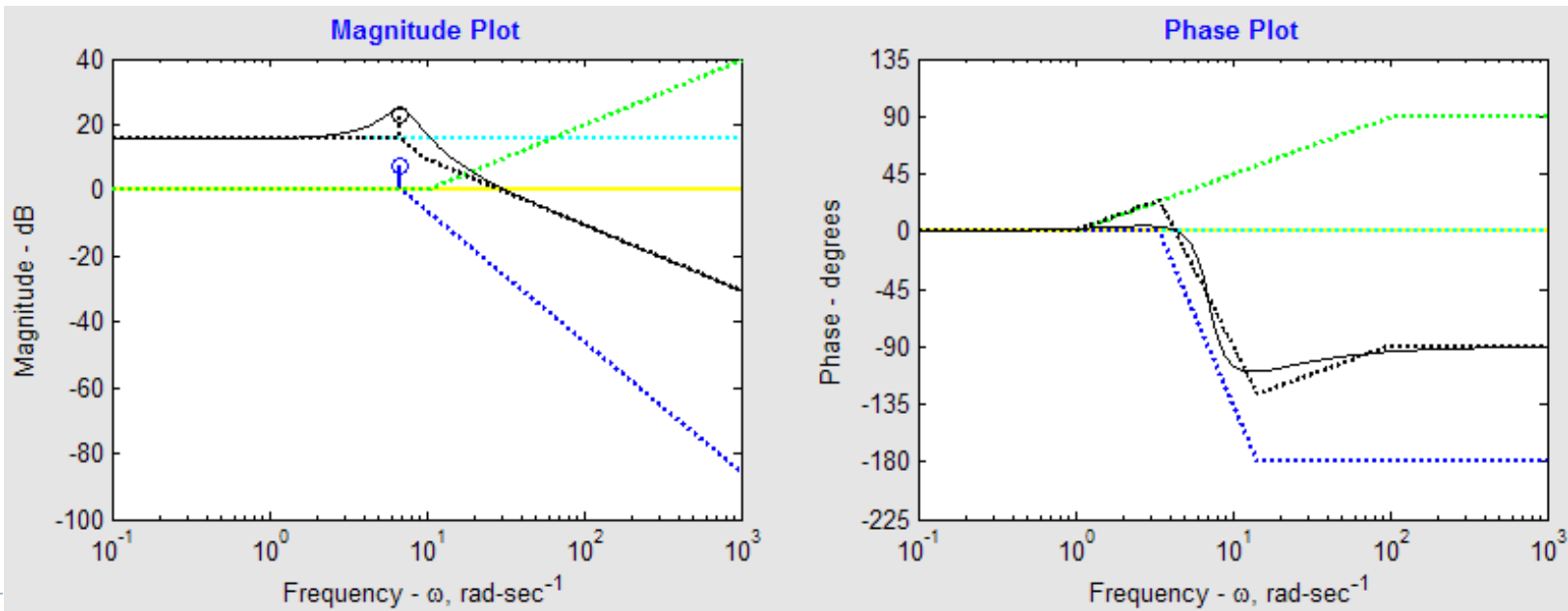
$$H_1(s) = 30 \frac{s + 10}{s^2 + 3s + 50}$$

$$H(s) = 30 \frac{s + 10}{s^2 + 3s + 50} = 30 \frac{10}{50} \frac{\frac{s}{10} + 1}{\frac{s^2}{50} + \frac{3}{50}s + 1} = 6 \frac{\frac{s}{10} + 1}{\frac{s^2}{50} + \frac{3}{50}s + 1}$$

- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 6 (16 dB)
- Complex Pole at $\omega_n=7.1$, $\zeta=0.21$ (-1.5 +/- 6.9j) Circle shows peak height.
- Real Zero at -10

- valor constante = 6,
 - um zero e $s=-10$,
 - e par de polos complexos conjugados em: raízes de: $s^2+3s+50=0$;
 - Polos complexos em $s=-1,5 \pm j6,9$ (onde $j=\sqrt{-1}$).
- Uma maneira mais comum (e útil para nossos propósitos) de expressar isso é usar a notação padrão para um polinômio de segunda ordem:

$$\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1 \quad \omega_0 = \sqrt{50} = 7.07, \quad \zeta = \frac{3\sqrt{50}}{2 \cdot 50} = 0.21$$



Problemas sugeridos:

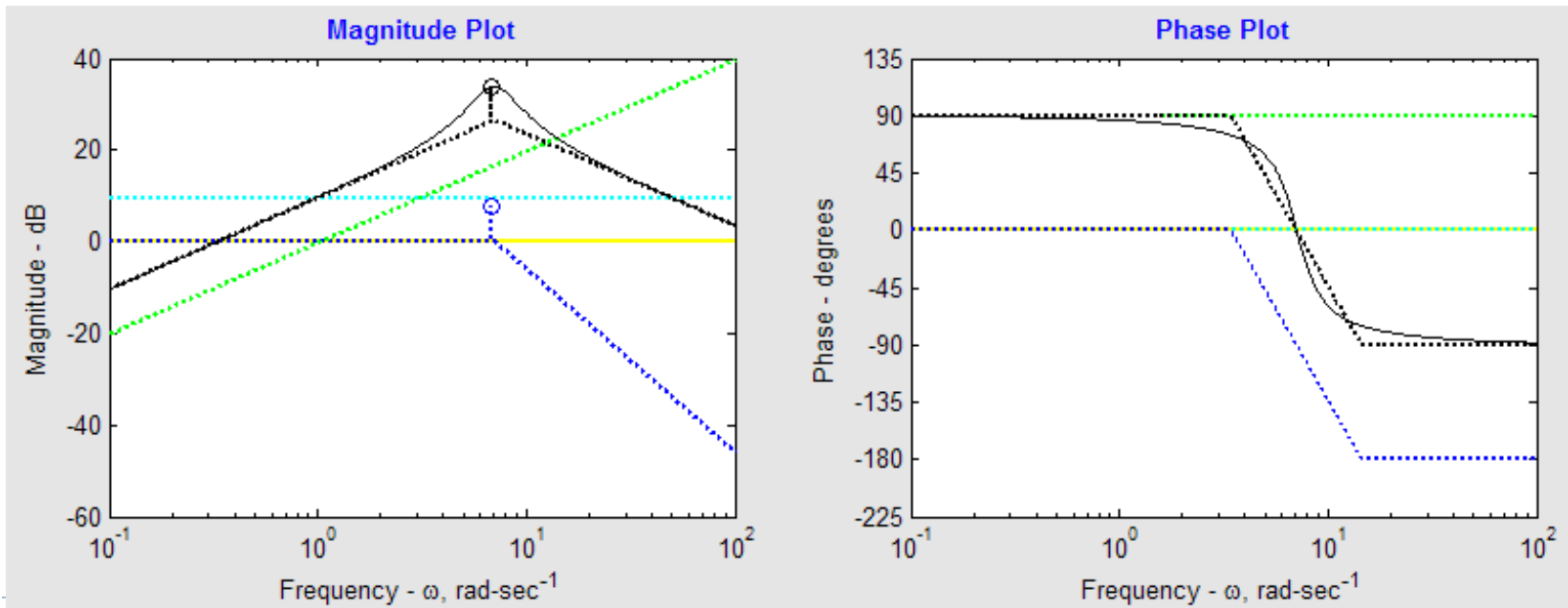
$$H_2(s) = 30 \frac{5s}{s^2 + 3s + 50}$$

$$H(s) = 30 \frac{s+10}{s^2 + 3s + 50} = 30 \frac{10}{50} \frac{\frac{s}{10} + 1}{\frac{s^2}{50} + \frac{3}{50}s + 1} = 6 \frac{\frac{s}{10} + 1}{\frac{s^2}{50} + \frac{3}{50}s + 1}$$

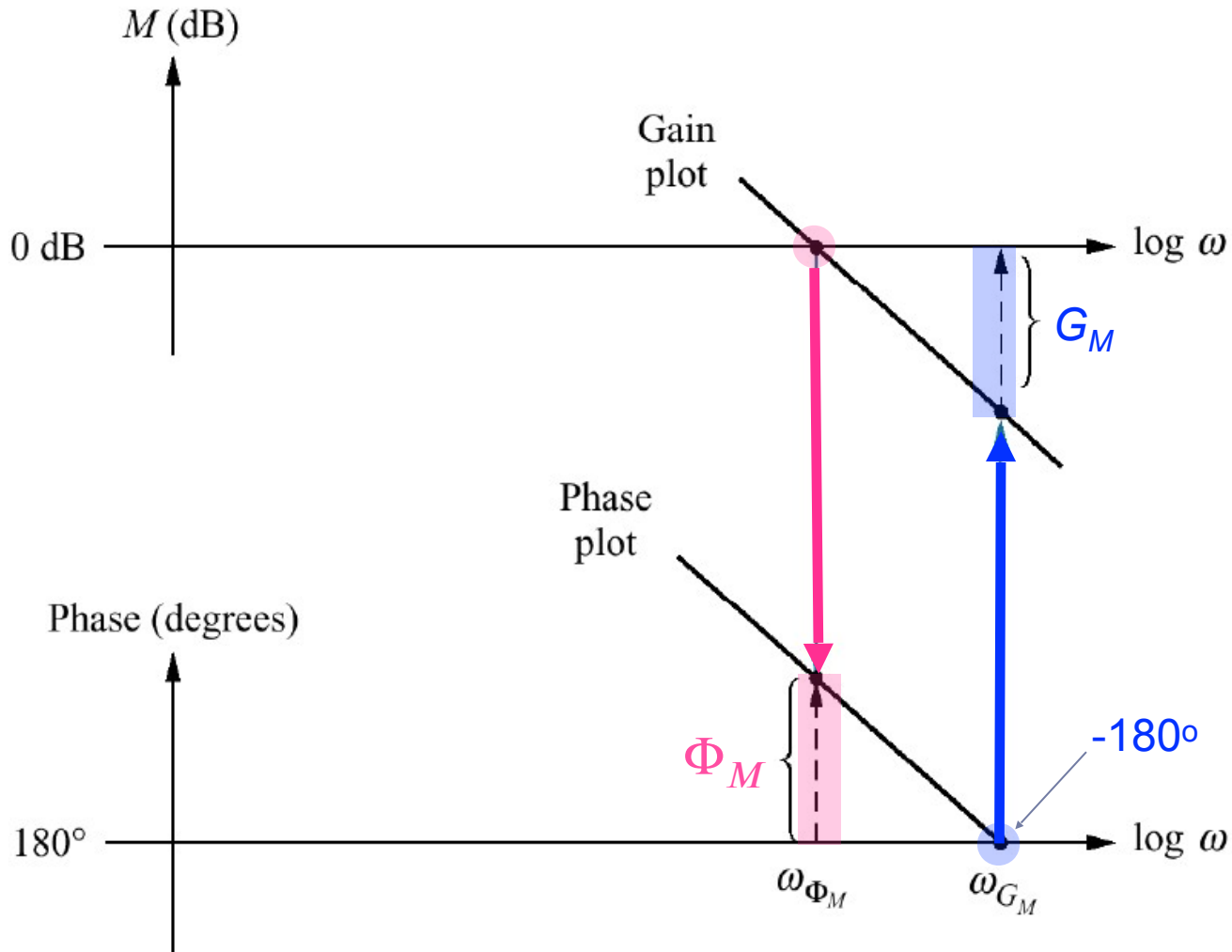
- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 3 (9.5 dB)
- Complex Pole at $\omega_n=7.1$, $\zeta=0.21$ (-1.5 +/- 6.9j) Circle shows peak height.
- Zero at origin

- valor constante = 6,
- um zero e $s=-10$,
- e par de polos complexos conjugados em: raízes de: $s^2+3s+50=0$, ou:

$$\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1 \quad \omega_0 = \sqrt{50} = 7.07, \quad \zeta = \frac{3\sqrt{50}}{2 \cdot 50} = 0.21$$

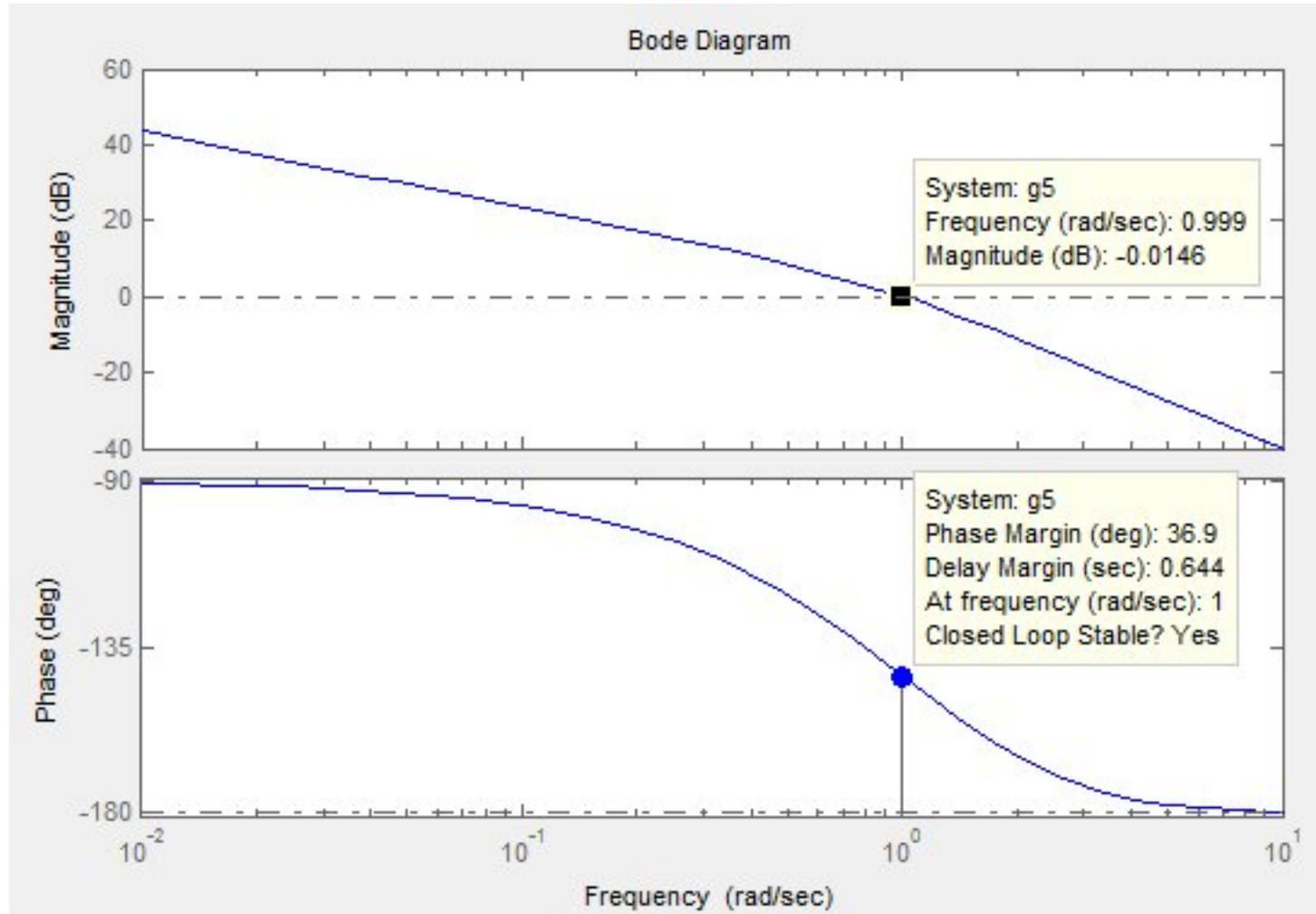


Estabilidade, Margem de Ganho, G_M e Margem de Fase, Φ_M através do Diagrama de Bode...



Outros Exemplos:

$$\begin{aligned} G_5(s) &= \frac{s+3}{s(s+1)(s+2)} \\ &= \frac{3\left(\frac{s}{3}+1\right)}{s(s+1)\cdot 2\left(\frac{s}{2}+1\right)} \\ &= \frac{3}{2} \cdot \frac{\left(\frac{s}{3}+1\right)}{s(s+1)\cdot\left(\frac{s}{2}+1\right)} \end{aligned}$$



Resumo:

Caso de pólo simples real...

$$G(s) = 1 + \frac{s}{\omega_p} = \frac{1}{1 + j\frac{\omega}{\omega_p}}$$

Onde ω_p = freq. de corte (-3 dB neste ponto);

Magnitude:

$$|G(j\omega)| = \left| \frac{1}{1 + j\frac{\omega}{\omega_p}} \right| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_p}\right)^2}}$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10} \left[\frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_p}\right)^2}} \right]$$

Quando $\omega \ll \omega_p$, teremos: $\omega/\omega_p \rightarrow 0$ e:

$$\sqrt{1 + (\omega/\omega_p)^2} \approx 1 \quad \therefore |G(j\omega)|_{dB} \approx -20 \log_{10}(1/1) = 0$$

Quando $\omega \gg \omega_p$, acontece: $\omega/\omega_p \rightarrow \infty$ e:

$$\sqrt{1 + (\omega/\omega_p)^2} \approx \sqrt{(\omega/\omega_p)^2} \approx \omega/\omega_p$$

$$|G(j\omega)|_{dB} \approx -20 \log_{10}(\omega/\omega_p)$$

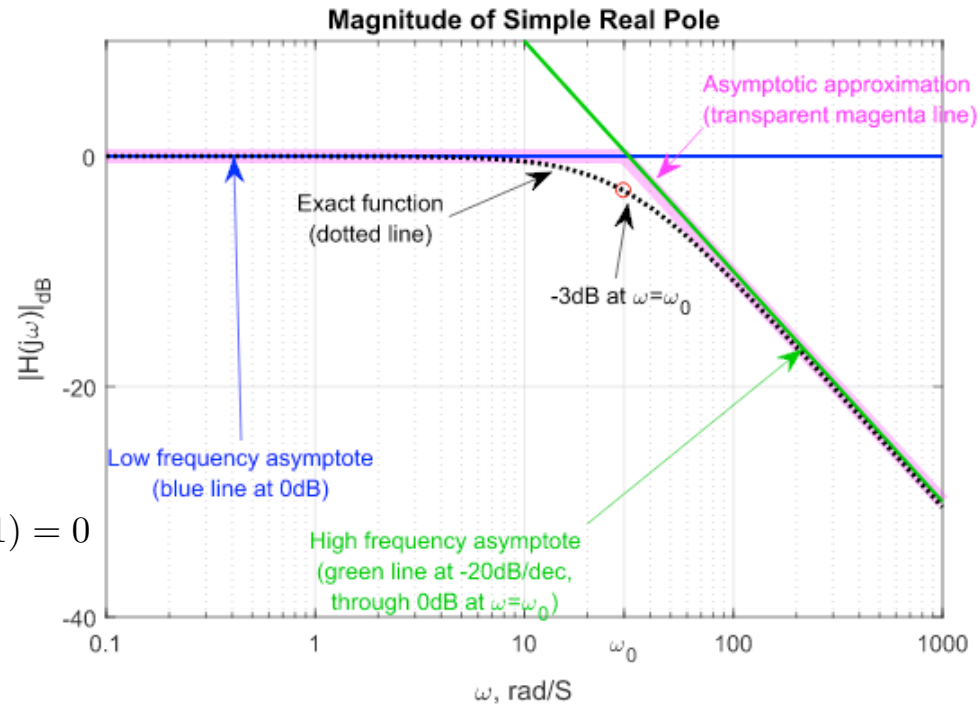
Se: $\omega = 10\omega_p \quad \therefore |G(j\omega)|_{dB} = -20 \log_{10}(10\omega_p/\omega_p)$

$$|G(j\omega)|_{dB} = -20 \log_{10}(10) = -20 \text{ dB}$$

Quando $\omega = \omega_p$:

$$|G(j\omega_p)|_{dB} = 20 \log_{10} \left(\frac{1}{\sqrt{1 + (\omega_p/\omega_p)^2}} \right)$$

$$|G(j\omega_p)|_{dB} = 20 \log_{10} (1/\sqrt{2}) \approx -3 \text{ dB}$$



Resumo:

Caso de pólo simples real...

$$G(s) = 1 + \frac{s}{\omega_p} = \frac{1}{1 + j\frac{\omega}{\omega_p}}$$

Onde ω_p = freq. de corte (-45° neste ponto);

Fase:

$$\angle G(j\omega) = \angle \left[\frac{1}{\left(1 + j\frac{\omega}{\omega_p}\right)} \right]$$

$$\angle G(j\omega) = -\angle \left(1 + j\frac{\omega}{\omega_p} \right) = -\tan^{-1} \left(\frac{\omega}{\omega_p} \right)$$

Quando $\omega \ll \omega_p$, teremos: $\omega/\omega_p \rightarrow 0$ e:

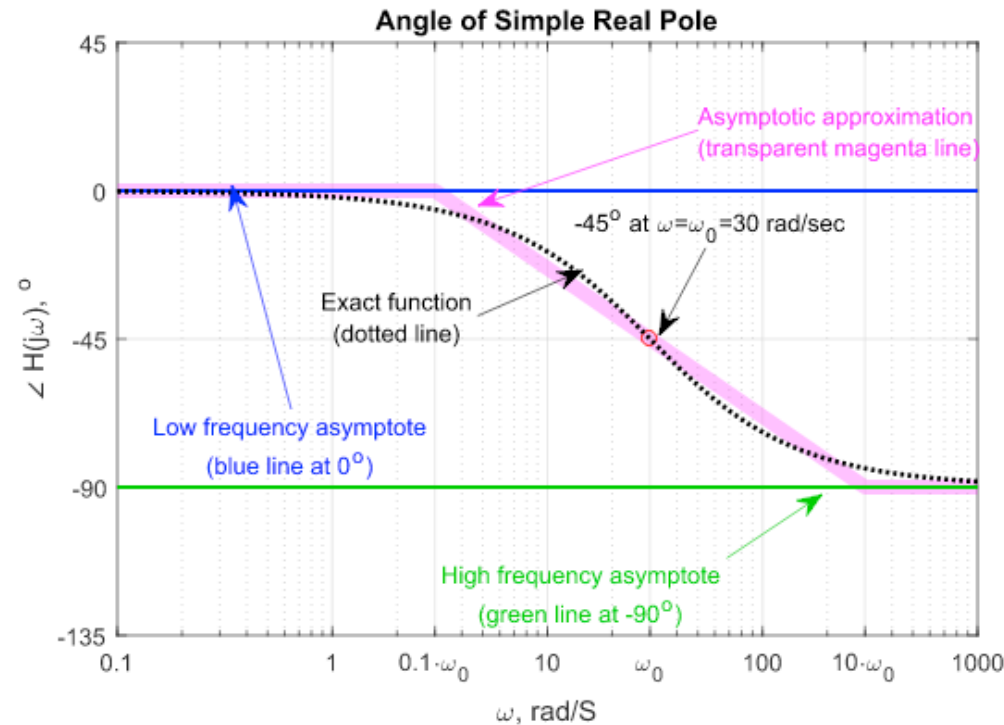
$$\angle G(j\omega) \approx -\tan^{-1}(0) = 0^\circ = 0 \text{ rad}$$

Quando $\omega \gg \omega_p$, acontece: $\omega/\omega_p \rightarrow \infty$ e:

$$\angle G(j\omega) \approx -\tan^{-1}(\infty) = -90^\circ = -\frac{\pi}{2} \text{ rad}$$

Quando $\omega = \omega_p$:

$$\angle G(j\omega) \approx -\tan^{-1}(1) = -45^\circ = -\frac{\pi}{4} \text{ rad}$$



Ref.: <https://lpsa.swarthmore.edu/Bode/BodeHow.html> (acessado Jun/2022)

Resumo:

Caso de zero simples real...

$$G(s) = 1 + \frac{s}{\omega_z} = 1 + j \frac{\omega}{\omega_z}$$

Onde ω_z = freq. de corte (+3 dB neste ponto);

Magnitude:

$$|G(j\omega)| = \left| 1 + j \frac{\omega}{\omega_z} \right|$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10} \left[\sqrt{1 + (\omega/\omega_z)^2} \right]$$

Quando $\omega \ll \omega_z$, teremos: $\omega/\omega_z \rightarrow 0$ e:

$$\sqrt{1 + (\omega/\omega_z)^2} \approx 1 \quad \therefore |G(j\omega)|_{dB} \approx 20 \log_{10}(1) = 0$$

Quando $\omega \gg \omega_z$, acontece: $\omega/\omega_z \rightarrow \infty$ e:

$$\sqrt{1 + (\omega/\omega_z)^2} \approx \sqrt{(\omega/\omega_z)^2} \approx \omega/\omega_z$$

$$|G(j\omega)|_{dB} \approx 20 \log_{10}(\omega/\omega_z)$$

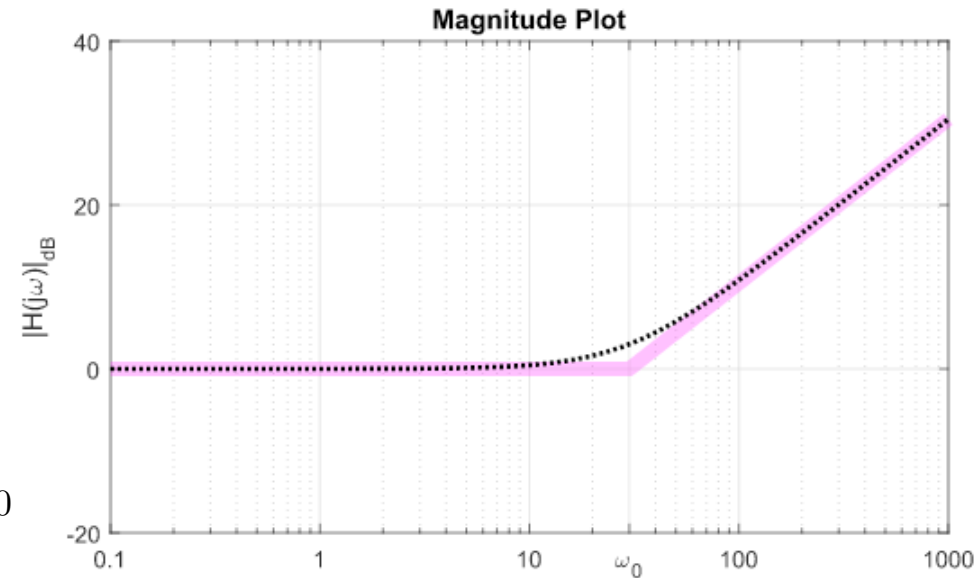
Se: $\omega = 10 \omega_z \quad \therefore |G(j\omega)|_{dB} = 20 \log_{10}(10 \cancel{\omega_z}/\cancel{\omega_z})$

$$|G(j\omega)|_{dB} = 20 \log_{10}(10) = 20 \text{ dB}$$

Quando $\omega = \omega_z$:

$$|G(j\omega_z)|_{dB} = 20 \log_{10} \left(\sqrt{1 + (\omega_z/\omega_z)^2} \right)$$

$$|G(j\omega_p)|_{dB} = 20 \log_{10} \left(\sqrt{2} \right) \approx 3 \text{ dB}$$



Resumo:

Caso de zero simples real...

$$G(s) = 1 + \frac{s}{\omega_z} = 1 + j \frac{\omega}{\omega_z}$$

Onde ω_z = freq. de corte (+45° neste ponto);

Fase:

$$\angle G(j\omega) = \angle (1 + j\omega/\omega_p)$$

$$\angle G(j\omega) = \angle \left(1 + j \frac{\omega}{\omega_p} \right) = \tan^{-1} \left(\frac{\omega}{\omega_p} \right)$$

Quando $\omega \ll \omega_z$, teremos: $\omega/\omega_z \rightarrow 0$ e:

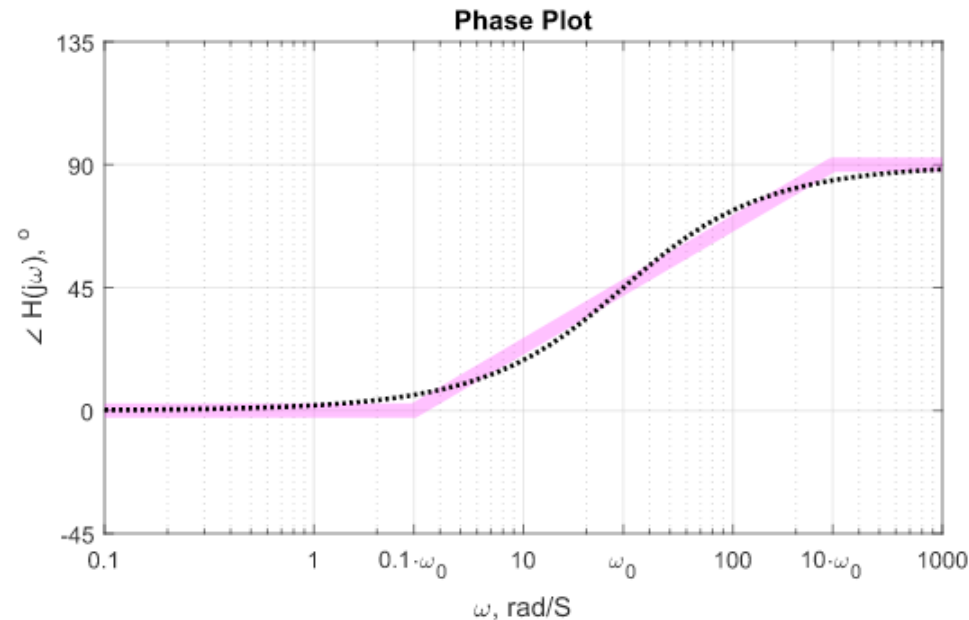
$$\angle G(j\omega) \approx \tan^{-1}(0) = 0^\circ = 0 \text{ rad}$$

Quando $\omega \gg \omega_z$, acontece: $\omega/\omega_z \rightarrow \infty$ e:

$$\angle G(j\omega) \approx \tan^{-1}(\infty) = 90^\circ = \frac{\pi}{2} \text{ rad}$$

Quando $\omega = \omega_z$:

$$\angle G(j\omega) \approx \tan^{-1}(1) = 45^\circ = \frac{\pi}{4} \text{ rad}$$



Ref.: <https://lpsa.swarthmore.edu/Bode/BodeHow.html> (acessado Jun/2022)

Resumo:

Pólo na origem

$$G(s) = \frac{1}{s} \quad \therefore \quad G(j\omega) = \frac{1}{j\omega} = -\frac{j}{\omega}$$

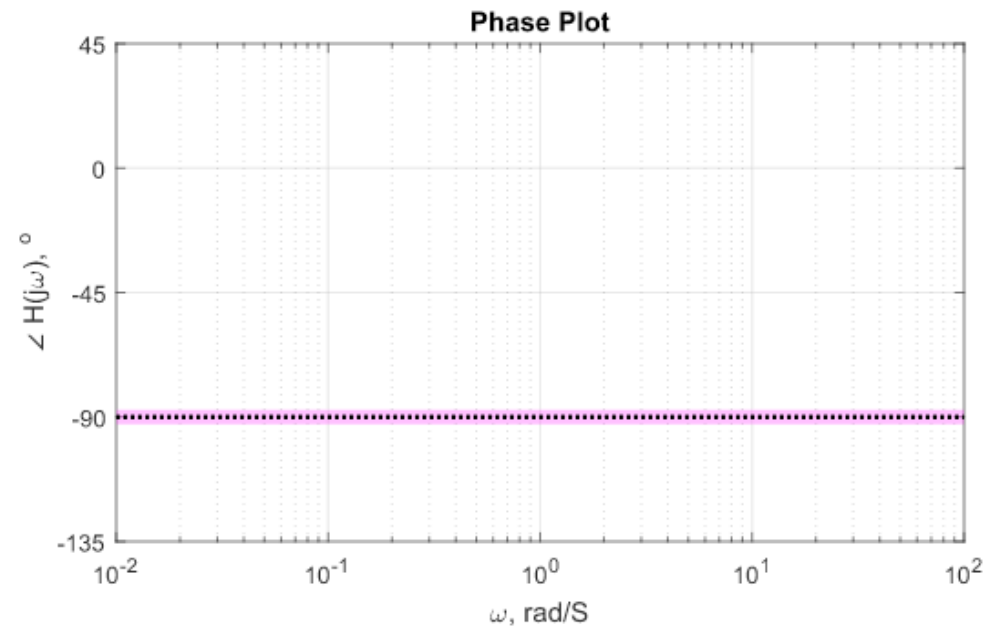
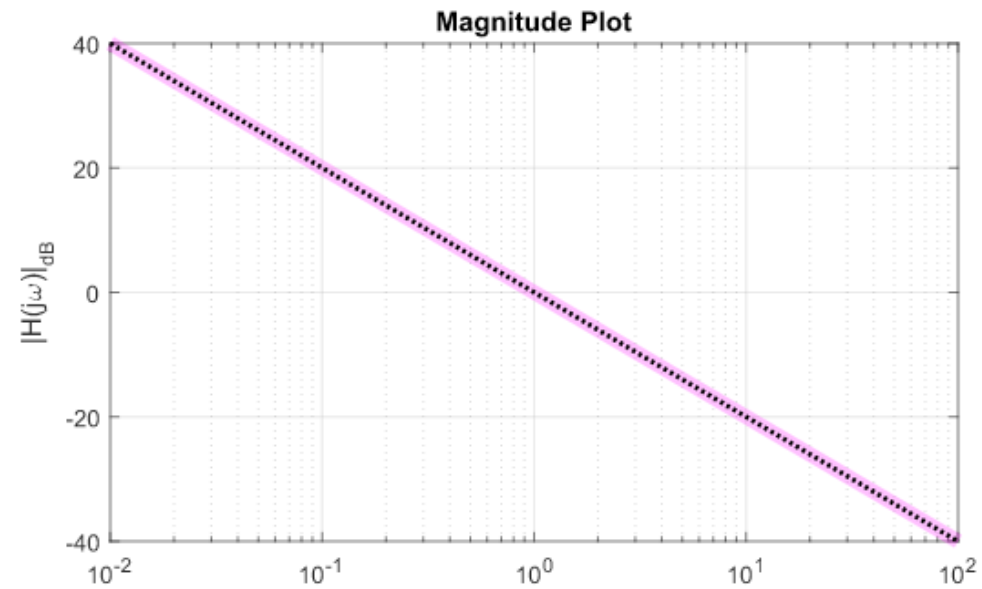
Magnitude:

$$|G(j\omega)| = \left| -\frac{j}{\omega} \right| = \frac{1}{\omega}$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10} \left(\frac{1}{\omega} \right) = -20 \log_{10}(\omega)$$

Fase:

$$\angle G(j\omega) = \angle \left(-\frac{j}{\omega} \right) = -90^\circ$$



Resumo:

Zero na origem

$$G(s) = s \quad \therefore \quad G(j\omega) = j\omega$$

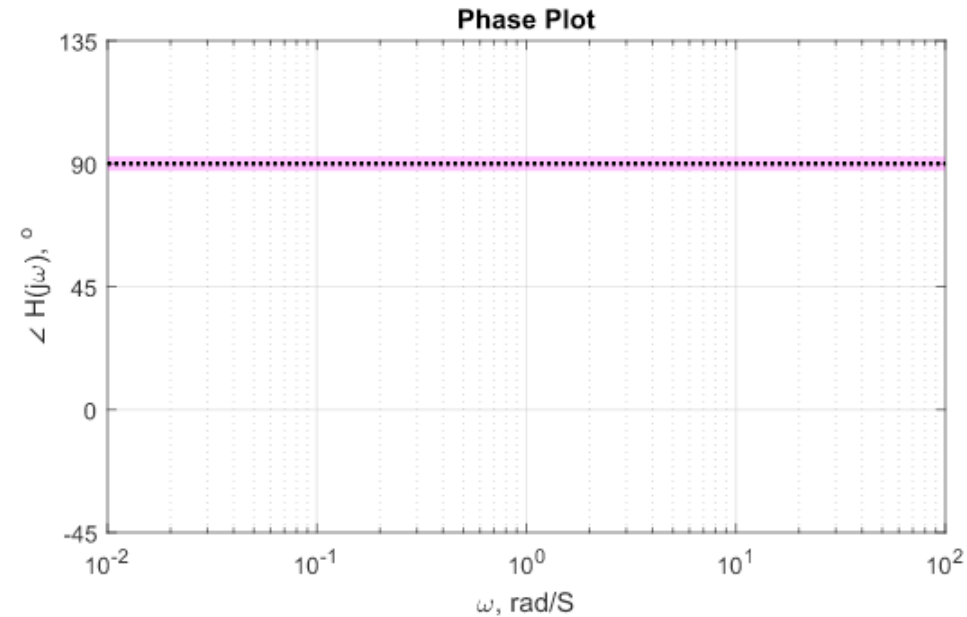
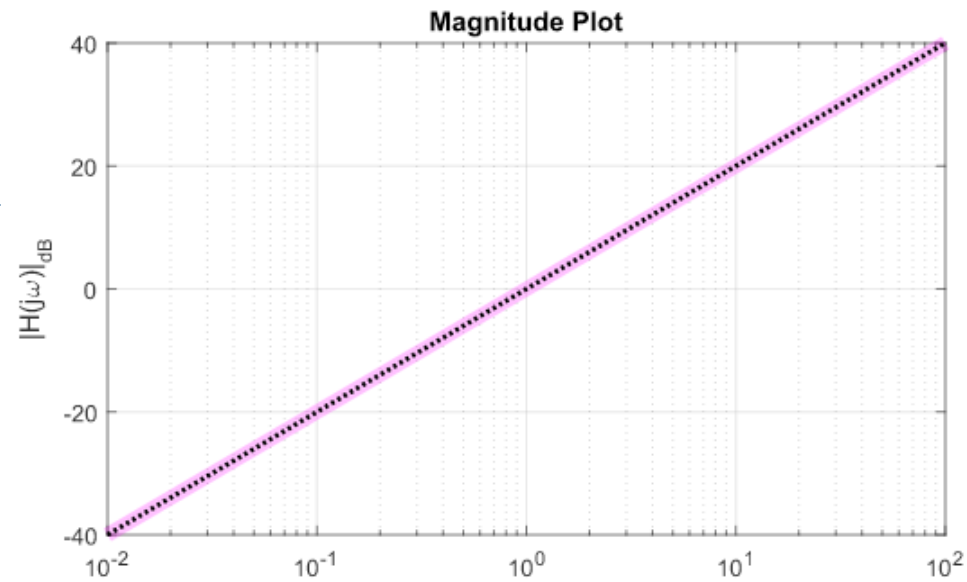
Magnitude:

$$|G(j\omega)| = |j\omega| = \omega$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10}(\omega) = 20 \log_{10}(\omega)$$

Fase:

$$\angle G(j\omega) = \angle (j\omega) = 90^\circ$$



Resumo: Pólos complexos

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + s2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

Magnitude:

$$|G(j\omega)| = \left| \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1} \right| = \left| \frac{1}{-\left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right) + 1} \right|$$

$$|G(j\omega)| = \left| \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + j\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]} \right| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$|G(j\omega)|_{dB} = -20 \cdot \log_{10} \left\{ \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} \right\} \quad (1)$$

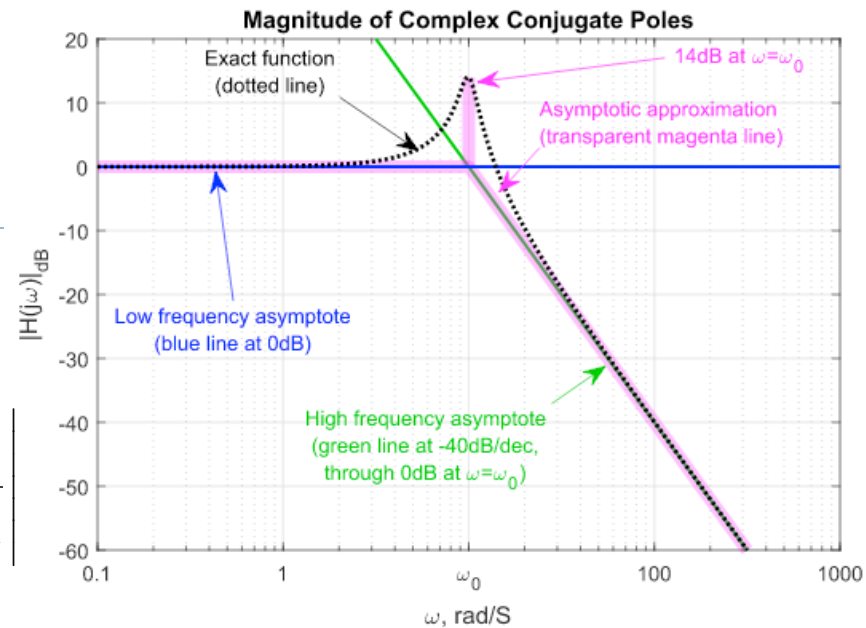
Caso de $\omega \ll \omega_n$: pode-se aproximar (1) para:

$$|G(j\omega)|_{dB} = -20 \cdot \log_{10}(1) = 0$$

Caso de $\omega \gg \omega_n$: pode-se aproximar (1) para:

$$|G(j\omega)|_{dB} = -20 \cdot \log_{10} \left[\left(\frac{\omega}{\omega_n}\right)^2 \right] = -40 \log_{10} \left(\frac{\omega}{\omega_n}\right)$$

Isto implica que para década acima de ω_n , magnitude = -40 (dB/déc).



Caso de $\omega \approx \omega_n$: ocorre um pico próximo de ω_n :

$$\text{Ocorre em } \omega_\tau: \omega_\tau = \omega_n \sqrt{1 - 2\zeta^2}$$

Pico ocorre para: $0 < \zeta < 1/2$, com valor de:

$$|G(j\omega_\tau)| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

Ver próximo slide para maiores detalhes.

Resumo

Pólos complexos ($0 < \zeta < 1$)

Seja:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + s2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

Um pico de ganho pode ser identificado em $|G(j\omega)|$, fazendo:

$$\left. \frac{d}{d\omega} (|G(j\omega)|) \right|_{\omega \rightarrow 0}$$

Este pico ocorre em: $\omega_\tau = \omega_n \sqrt{1 - 2\zeta^2}$,

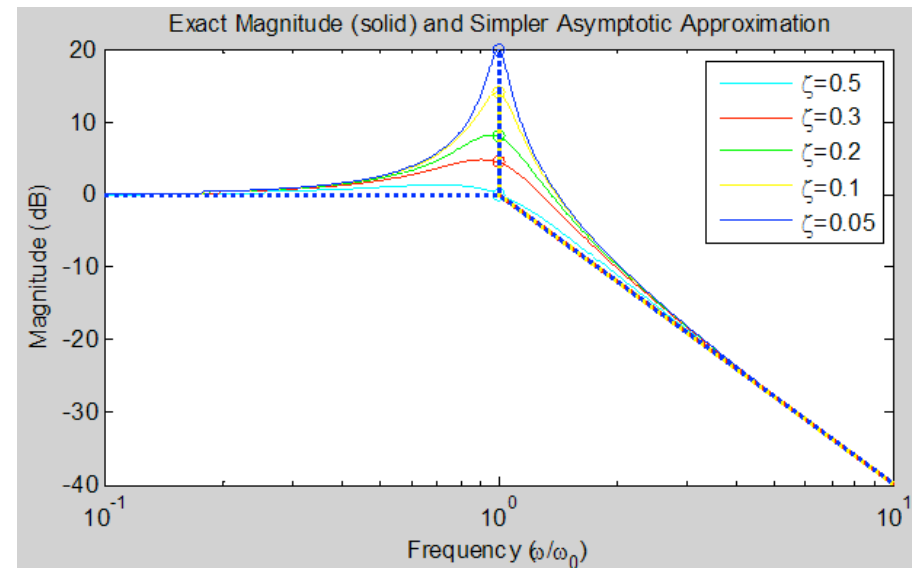
Com magnitude de:

$$|G(j\omega)| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \approx \frac{1}{2\zeta}$$

$$|G(j\omega)| = -20 \log_{10} \left(2\zeta\sqrt{1 - \zeta^2} \right)$$

O pico apenas existe para: $0 < \zeta < 0,707 (= 1/\sqrt{2})$

ζ	Peak frequency			Peak Height		
	Exact	Approx	% diff	Exact	Approx	% diff
	$\omega_0 \sqrt{1 - 2\zeta^2}$	ω_0		$\frac{1}{2\zeta\sqrt{1 - \zeta^2}}$	$\frac{1}{2\zeta}$	
0.5	$0.71\omega_0$	ω_0	29%	1.15	1.00	15%
0.4	$0.83\omega_0$	ω_0	17%	1.37	1.25	9.1%
0.3	$0.91\omega_0$	ω_0	9.5%	1.75	1.67	4.8%
0.2	$0.96\omega_0$	ω_0	4.1%	2.55	2.50	2.1%
0.1	$0.99\omega_0$	ω_0	1.0%	5.02	5.00	0.5%
0.05	$1.00\omega_0$	ω_0	0.3%	10.0	10.0	0.1%



Ref.: <https://lpsa.swarthmore.edu/Bode/underdamped/underdampedApprox.html#Simpler> (acessado em Jun/2022)

Resumo: Pólos complexos

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + s2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

Fase:

$$\angle G(j\omega) = \angle \left[\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1} \right]$$

$$\angle G(j\omega) = -\angle \left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1 \right]$$

$$\angle G(j\omega) = -\angle \left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) \right]$$

$$\angle G(j\omega) = -\tan^{-1} \left[\frac{2\zeta \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right] \quad (2)$$

Caso de $\omega \ll \omega_n$: pode-se aproximar (2) para:

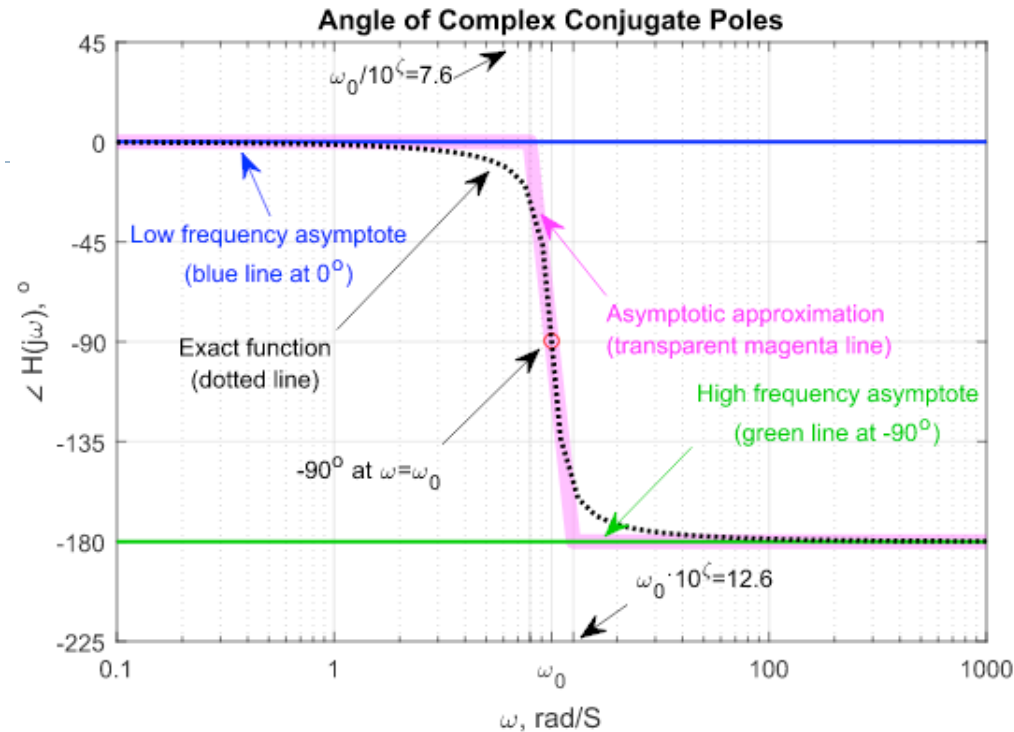
$$\angle G(j\omega) \approx -\tan^{-1} \left(\frac{2\zeta\omega}{\omega_n} \right) \approx -\tan^{-1}(0) = 0^\circ = 0 \text{ rad}$$

Caso de $\omega \gg \omega_n$: pode-se aproximar (2) para:

$$\angle G(j\omega) \approx -180^\circ = -\pi \text{ rad}$$

Caso de $\omega = \omega_n$:

$$\angle G(j\omega) \approx -90^\circ = -\pi/2 \text{ rad}$$



Resumo

Pólos complexos ($0 < \zeta < 1$)

Seja:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + s2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

$$\angle G(j\omega) = -\tan^{-1} \left[\left(2\zeta \frac{\omega}{\omega_n}\right) / \left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right) \right] \quad (2)$$

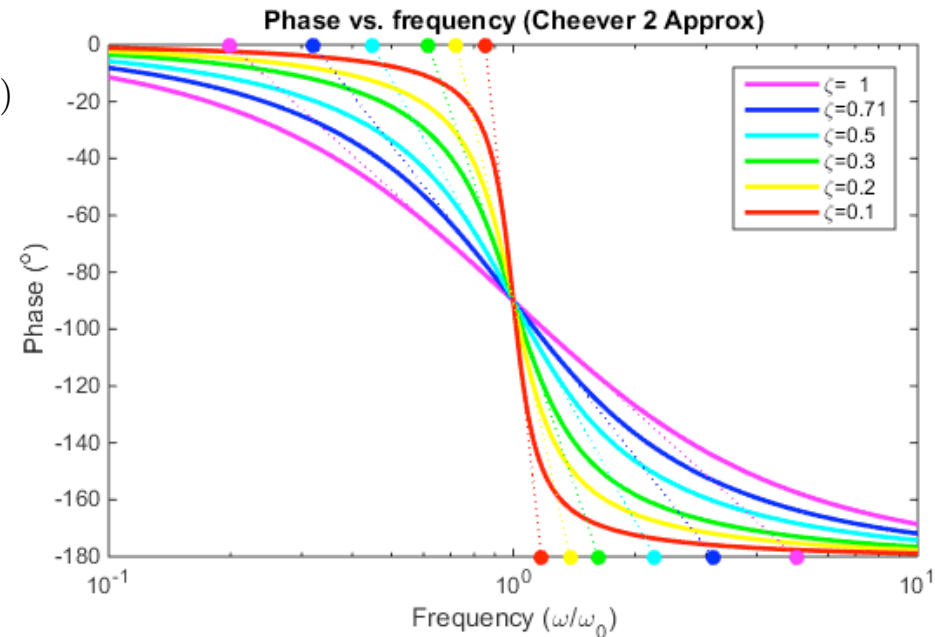
A declividade de $\angle G(j\omega)$, pode ser obtida, fazendo:

$$\left. \frac{d}{d\omega} (\angle G(j\omega)) \right|_{\omega \rightarrow 0}$$

O que permite identificar os pontos:

$$\omega_{low} = \omega_n e^{-\frac{\pi}{2}\zeta} \approx \omega_n \cdot 0,2^\zeta$$

$$\omega_{high} = \frac{\omega_n}{e^{-\frac{\pi}{2}\zeta}} \approx \frac{\omega_n}{0,2^\zeta} = \omega \cdot 5^\zeta$$



Ref.: <https://lpsa.swarthmore.edu/Bode/underdamped/underdampedApprox.html#Simpler> (acessado em Jun/2022)

Resumo: Esboços Diagramas de Bode

► <https://lpsa.swarthmore.edu/Bode/BodeHow.html>

The Asymptotic Bode Diagram: Deri X +

← → ↻ <https://lpsa.swarthmore.edu/Bode/BodeHow.html> ☆ ⌵ ⌵ ⌵

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Interactive Demo: Bode Plot of a Pair of Complex Conjugate Poles

This demonstration shows how a second order pole (complex conjugate roots) expressed as:

$$H(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1} = \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j2\zeta\frac{\omega}{\omega_0}}$$

is displayed on a Bode plot. You can change ω_0 , and ζ . The value of ω_0 is constrained such that $0.1 \leq \omega_0 \leq 10$ rad/second, and $0.05 \leq \zeta \leq 0.99$.

Enter value for ω_0 : or click and drag on graph to set ω_0 , and use text-box or slider, below, for ζ .

ζ 0.99

Asymptotic Magnitude: The asymptotic magnitude plot starts (at low frequencies) at 0 dB and stays at that level until it gets to ω_0 . At that point the gain starts dropping with a slope of -40 dB/decade. *Note: it is -40 dB per decade because there are two poles in the denominator.*

If $\zeta < 0.5$ we estimate the peak height as $|H(j\omega_{peak})| \approx \frac{1}{2\zeta}$ (exact height is $|H(j\omega_{peak})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$). We approximate the peak location at $\omega_{peak} \approx \omega_0$ (exact peak location is at $\omega_{peak} = \omega_0\sqrt{1-2\zeta^2}$).

However, if $\zeta \geq 0.5$, the peak is sufficiently small that we don't include it in our plot.

Since $\zeta < 0.5$, we draw a peak. Note how close together the approximate and exact values are for ω_{peak} and $|H(j\omega_{peak})|$.

	ω_{peak}	$ H(j\omega_{peak}) $	$ H(j\omega_{peak}) _{dB}$
Approximate	1.00	2.50	7.96
Exact	0.96	2.55	8.14

Magnitude

Phase

— phsApprox - - - phsExact - - - w0
- - - w0_low - - - w0_high

Resposta frequencial:: Aplicação

► <https://lpsa.swarthmore.edu/Bode/BodeWhat.html>

Interactive Demo
Choose a transfer function.

$H(s) = \frac{1}{1+2s}$ $H(j\omega) = \frac{1}{1+j2\omega}$

$H(s) = \frac{1.6}{s^2+0.5s+1.6}$ $H(j\omega) = \frac{1.6}{(1.6-\omega^2)+j(0.5\omega)}$

Set input parameters, $V_{in}(t)=A\cos(\omega t+\phi)$.

Set ω : ω 0 3

Set A: A 0.2 2

Set ϕ : ϕ -180 180

At $\omega = 1$, $H(j\omega) = 1/(1.00 + j2.00) = 0.45\angle -63.4^\circ = M\angle\theta$.
Since the input can be represented as $1.5\angle 0^\circ$,
The output is $M \cdot A\angle(\theta+\phi) = 0.68\angle -63.4^\circ$.

	Magnitude	Phase	Time Domain
$H(j\omega)$	0.45	-63.4°	$0.45 \cdot \cos(1 \cdot t + -63.4^\circ)$
Input	1.5	0°	$1.5 \cdot \cos(1 \cdot t + 0^\circ)$
Output	0.68	-63.4°	$0.68 \cdot \cos(1 \cdot t + -63.4^\circ)$

Directions for Use

Use the radio buttons to choose a transfer function, and the sliders to choose the frequency, amplitude and phase of the input (you can also set frequency by clicking and dragging in either of the top two graphs.)

The paragraph below the sliders goes through the calculation of the numerical value of the transfer function at the chosen frequency, and gives $H(j\omega)$ in terms of magnitude and phase. Note that these are also shown on the top two graphs by a dot. To find the

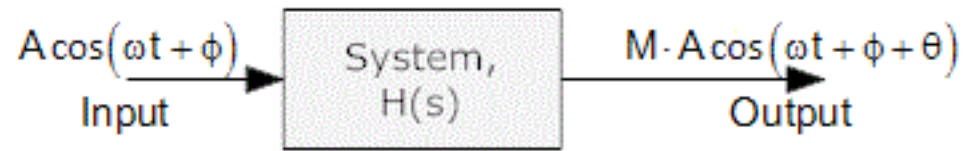
Magnitude of $H(j\omega)$, (i.e., $|H(j\omega)|$), vs ω

Phase of $H(j\omega)$, (i.e., $\angle H(j\omega)$), vs ω

$V_{in}(t)$ & $V_{out}(t)$ vs. t

Resposta frequencial

Aplicação



- ▶ Problema: mostre o sinal de saída para o sistema ao lado quando se aplica ao mesmo um cosseno de 2 Vpp à frequência de 1 Hz.
- ▶ Considerar: $R = 2 \text{ M}\Omega$ e $C = 1 \text{ }\mu\text{F}$
- ▶ Solução:

$$H(s) = \frac{1}{1 + sRC} = \frac{1}{1 + 2 \times 10^6 \cdot 1 \times 10^{-6} \cdot s} = \frac{1}{1 + 2s}$$

$$H(j\omega) = \frac{1}{1 + j2\omega} = \frac{1}{1 + j\frac{\omega}{\omega_p}}$$

$$|H(j\omega)| = \left| \frac{1}{1 + j\frac{\omega}{\omega_p}} \right| = \frac{1}{\sqrt{1^2 + \left(\frac{\omega}{\omega_p}\right)^2}}$$

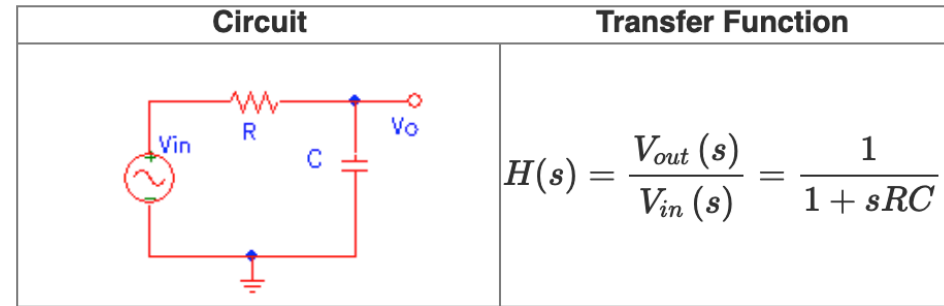
$$\angle H(j\omega) = -\angle \left(1 + j\frac{\omega}{\omega_p} \right) = -\tan^{-1} \left(\frac{\omega}{\omega_p} \right)$$

$$\omega_p = 1/2 \text{ (rad/s)}$$

$$\therefore f_p = \frac{\omega}{2\pi} = \frac{1}{4\pi} = 0,079577 \text{ Hz}$$

$$V_{out}(j\omega) = V_{in}(j\omega) \cdot H(j\omega)$$

$$|V_{out}(j\omega)| =$$



Usando Matlab:

```
>> H=tf(1,[2 1])
```

```
H =
```

```
1
```

```
-----  
2 s + 1
```

```
>> pole(H)
```

```
-0.5
```

```
>> f_p=1/(4*pi)
```

```
f_p =
```

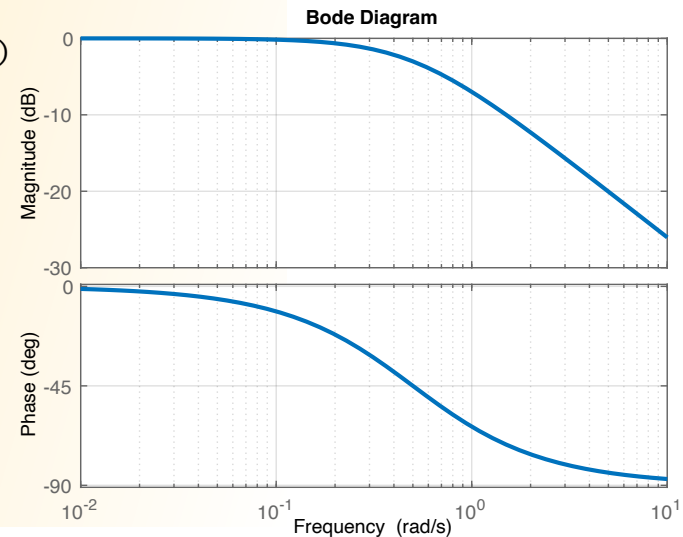
```
0.079577
```

```
>> zpk(H)
```

```
0.5
```

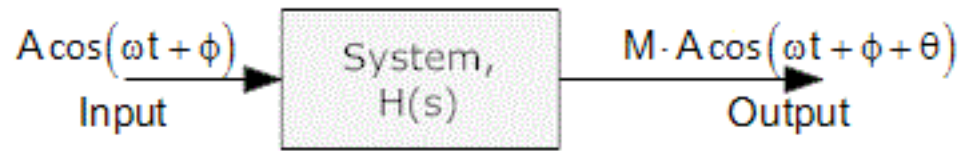
```
-----  
(s+0.5)
```

```
>> bode(G)
```



Resposta frequencial

Aplicação



- ▶ Problema: mostre o sinal de saída para o sistema ao lado quando se aplica ao mesmo um cosseno de 2 Vpp à frequência de 1 Hz.
- ▶ Considerar: $R = 2 \text{ M}\Omega$ e $C = 1 \text{ }\mu\text{F}$
- ▶ Solução:

$$H(s) = \frac{1}{1 + sRC} = \frac{1}{1 + 2 \times 10^6 \cdot 1 \times 10^{-6} \cdot s} = \frac{1}{1 + 2s}$$

$$H(j\omega) = \frac{1}{1 + j2\omega} = \frac{1}{1 + j\frac{\omega}{\omega_p}}$$

$$|H(j\omega)| = \left| \frac{1}{1 + j\frac{\omega}{\omega_p}} \right| = \frac{1}{\sqrt{1^2 + \left(\frac{\omega}{\omega_p}\right)^2}}$$

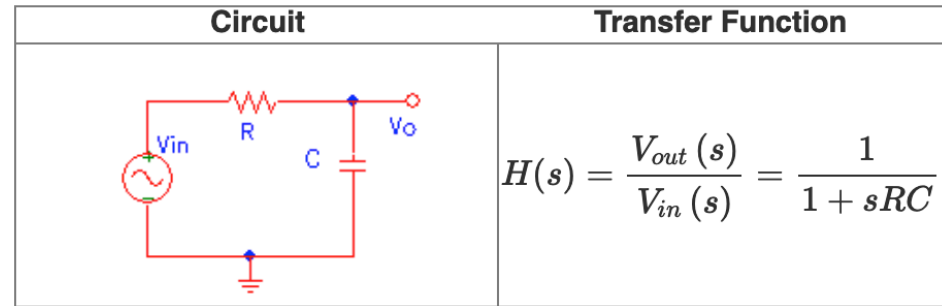
$$\angle H(j\omega) = -\angle \left(1 + j\frac{\omega}{\omega_p} \right) = -\tan^{-1} \left(\frac{\omega}{\omega_p} \right)$$

$$\omega_p = 1/2 \text{ (rad/s)}$$

$$\therefore f_p = \frac{\omega}{2\pi} = \frac{1}{4\pi} = 0,079577 \text{ Hz}$$

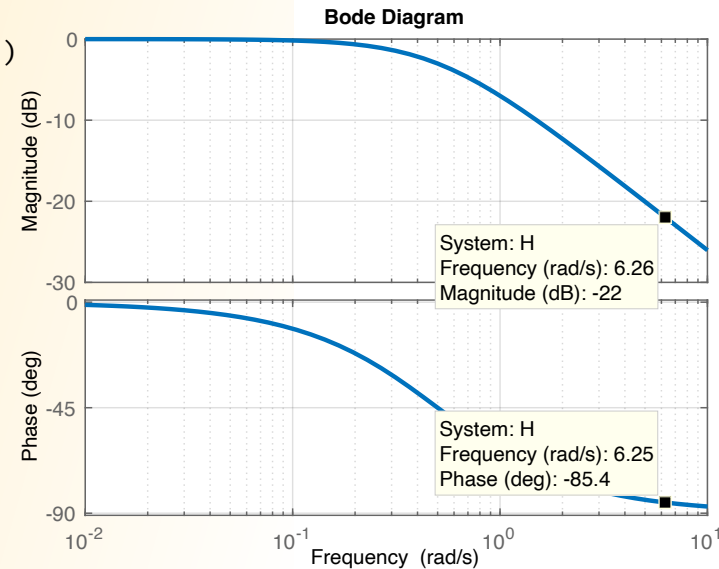
$$V_{out}(j\omega) = V_{in}(j\omega) \cdot H(j\omega)$$

$$|V_{out}(j\omega)| =$$



Usando Matlab:

```
>> H=tf(1,[2 1])
H =
    1
    -----
    2 s + 1
>> pole(H)
    -0.5
>> f_p=1/(4*pi)
f_p =
    0.079577
>> zpk(H)
    0.5
    -----
    (s+0.5)
>> bode(G)
>> f=2*pi
f =
    6.2832
```



Resposta frequencial

Aplicação

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$$H(j\omega) = \frac{1}{1 + j2\omega} = \frac{1}{1 + j\frac{\omega}{\omega_p}}$$

Ganho em $\omega = 1 \text{ Hz} = 2\pi \text{ rad/s}$:

$$|G(j\omega)|_{dB} = -22 \text{ dB}$$

$$-22 = 20 \log_{10}(|G(j\omega)|)$$

$$|G(j\omega)| = 10^{-\frac{22}{20}} = 0,079433$$

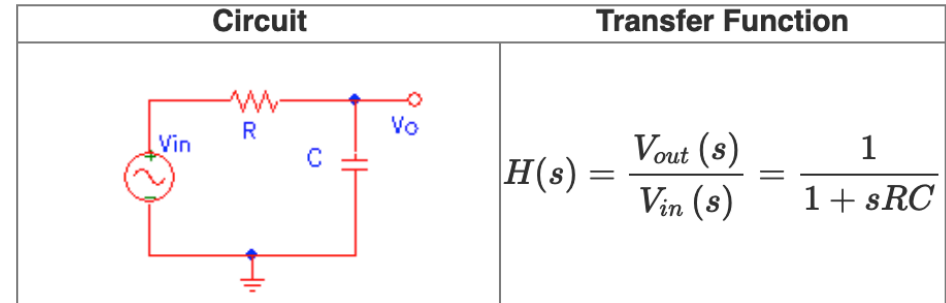
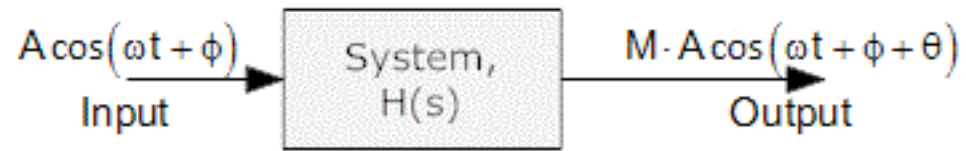
$$\angle G(j\omega) = -85,4^\circ = -85,4^\circ \cdot \frac{\pi}{180^\circ} = -1.4905 \text{ rad}$$

$$T \rightarrow 360^\circ (2\pi)$$

$$\Delta t \leftarrow 85,4^\circ$$

$$\Delta t = \frac{85,4^\circ \cdot T}{360^\circ} = 0,037755 \text{ segundos}$$

Considerando $T = 0,15915$ segundos.



Usando Matlab:

```
>> H=tf(1,[2 1])
```

```
H =
```

```
1
```

```
-----
```

```
2 s + 1
```

```
>> pole(H)
```

```
-0.5
```

```
>> f_p=1/(4*pi)
```

```
f_p =
```

```
0.079577
```

```
>> zpk(H)
```

```
0.5
```

```
-----
```

```
(s+0.5)
```

```
>> bode(G)
```

```
>> rad=2*pi*1
```

```
rad =
```

```
6.2832
```

```
>> ganho=10^(-22/20)
```

```
ganho =
```

```
0.079433
```

```
>> T=1/1
```

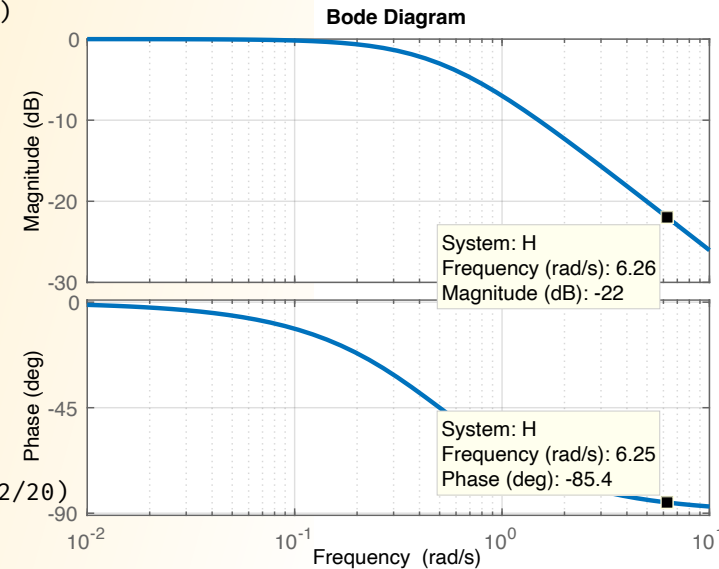
```
T =
```

```
1.0
```

```
>> Delta_t=(85.4*T)/360
```

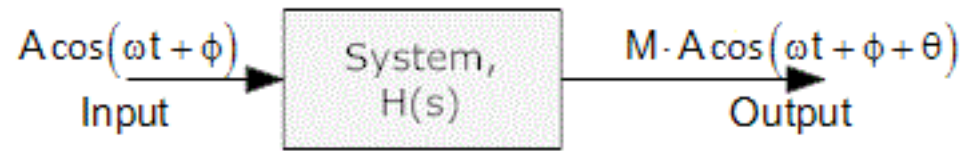
```
Delta_t =
```

```
0.23722
```

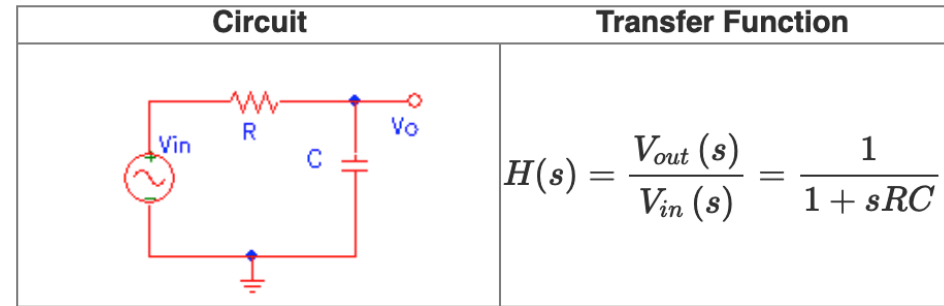


Resposta frequencial

Aplicação

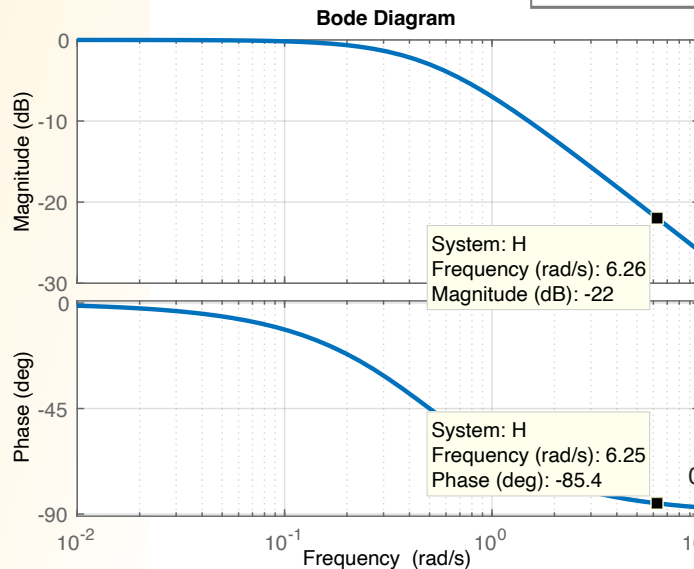


- ▶ Problema: mostre o sinal de saída para o sistema ao lado quando se aplica ao mesmo um cosseno de 2 Vpp à frequência de 1 Hz.
- ▶ Considerar: $R = 2 \text{ M}\Omega$ e $C = 1 \text{ }\mu\text{F}$
- ▶ Solução:



```

Usando Matlab:
>> H=tf(1,[2 1])
H =
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      1
    2 s + 1
>> pole(H)
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>> f_p=1/(4*pi)
f_p =
    0.079577
>> zpk(H)
      0.5
-----
    (s+0.5)
>> bode(G)
>> rad=2*pi*1
rad =
    6.2832
>> ganho=10^(-22/20)
ganho =
    0.079433
>> T=1/1
T =
    1.0
>> Delta_t=(85.4*T)/360
Delta_t =
    0.23722
    
```



```

>> rad=(85.4*pi)/180
rad =
    1.4905
>> ezplot(@t)1*cos(2*pi*1*t),[0 2])
% onda entrada
>> hold on
>> ezplot(@t)ganho*cos(2*pi*1*t-
rad),[0 2]) // onda saída
>> axis([0 2 -1 1])
>> grid
>> legend('V_{in}','V_{out}')
    
```

