Transformada Z
"Revisão Rápida"

Prof. Fernando Passold

Transformada Z ➠ Definição

$$f^*(t) = \sum_{k=0}^{\infty} f(kT)\cdot\delta(t-kT) = f(t)\cdot\delta_T(t)$$

Sinal amostrado

$$\mathscr{L}\{f^*(t)\} = F^*(s) = \sum_{k=0}^{\infty} f[kT]\cdot e^{-kTs}$$

Definição: $z = e^{T\,s}$ $\qquad s = \dfrac{1}{T}\ln z$

$$F(z) = \sum_{k=0}^{\infty} f(kT)\,z^{-k}$$

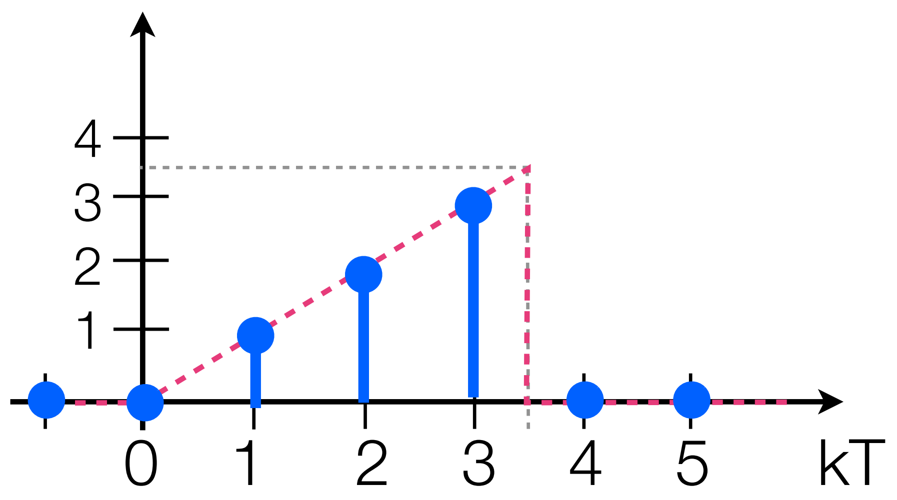
$\lim_{T\to 0} F(z) \neq F(s)$

Detalhes

$\lim_{T\to 0} f^*(t) = f(t)$

$$\mathcal{Z}\left\{f(kT)\right\} = F(z) = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

$$F(z) = f(0) + f(T)z^{-1} + f(2T)z^{-2} + \ldots + f(kT)z^{-k} + \ldots$$



$$f(t) = \begin{cases} 0, & \forall \quad t < 0 \\ t, & \forall \quad 0 \leq t < 3,5 \\ 0, & \forall \quad t \geq 3,5 \end{cases}$$

$$F(z) = 0z^0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + 0\cdot z^{-4} + \ldots$$

$$\begin{array}{rcl} f(0) & = & 0 \\ f(1) & = & 1 \\ f(2) & = & 2 \\ f(3) & = & 3 \\ f(4) & = & 0 \\ & \vdots & \\ f(nT) & = & 0, n > 3 \end{array}$$

Algumas transformadas Z

Função Impulso: $\mathcal{Z}\left\{\delta(t)\right\} = \mathcal{Z}\left\{\delta(kT)\right\} = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1$

Algumas transformadas Z

Função Degrau: $\mathcal{Z}\left\{u(t)\right\} = \mathcal{Z}\left\{u^*(T)\right\} = \sum_{k=0}^{\infty} 1 \cdot z^{-k} = \quad ?$

Série Geométrica: $S(q) = a + aq + aq^2 + \ldots + aq^n = \sum_{k=0}^{n} a \cdot q^k$

a = 1º termo da série e *q* = razão da série.

$$\sum_{k=0}^{n} a \cdot q^k = \frac{a - a\,q^{n+1}}{1-q}$$

$$\mathcal{Z}\left\{u(t)\right\} = \lim_{n\to\infty} \left.\frac{1 - 1(z^{-1})^{n+1}}{1 - z^{-1}}\right|_{0}^{\infty} = \lim_{n\to\infty} \frac{1 - \overbrace{z^{-n}}^{=0} \cdot 1}{1 - z^{-1}} = \frac{1}{1 - z^{-1}}$$

Generalizando:

$$\sum_{n=0}^{\infty} Ax^n = \frac{A}{1-x}, \text{ se } |x| < 1$$

Notas detalhes:

A função x^n pode convergir ou não:

Se x<1 teremos:
Ex.:

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0{,}25$$

$$\left(\frac{1}{2}\right)^3 = \frac{1}{8} = 0{,}125$$

Ou seja, uma função decrescente!
(converge)

Se x>1 teremos:
Ex.:

$$(2)^2 = 4$$

$$(2)^3 = 8$$

Ou seja, uma função crescente!
(não converge)

Algumas transformadas Z

Sequência geométrica: $f[kT] = a^k, \quad \forall\, k = 0,1,2,3,\ldots$

$$f[kT] = \{1, a, a^2, a^3, \ldots\}$$

$$F(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n = 1 + a z^{-1} + a^2 z^{-2} + \ldots$$

$$F(z) = \frac{1}{1 - a z^{-1}} \cdot \frac{z^{+1}}{z^{+1}} = \frac{z}{z - a}$$

$$\mathrm{Z}\{a^k\} = \frac{z}{z - a}$$

Série Geométrica: $S(q) = a + aq + aq^2 + \ldots + aq^n = \sum_{k=0}^{n} a \cdot q^k$

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Generalizando:

$$\sum_{n=0}^{\infty} A x^n = \frac{A}{1 - x}, \text{ se } |x| < 1$$

Repare no seguinte:

A função $F(z) = \dfrac{z}{z - a}$ pode convergir ou não:

Se *a* > 1 teremos:

Ex.: $F(z) = \dfrac{z}{z - 2}$

$$f[kT] = \sum_{k=0}^{\infty} (2)^k z^{-k} = 1 + 2\,z^{-1} + 4\,z^{-2} + 8\,z^{-3} + \ldots$$

Ou seja, uma função crescente! (NÃO converge)

Se *a* < 1 teremos:

Ex.: $F(z) = \dfrac{z}{z - 0{,}5}$

$$f[kT] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^{-k} = 1 + \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2} + \frac{1}{8} z^{-3} + \ldots$$

Ou seja, uma função decrescente! (converge)

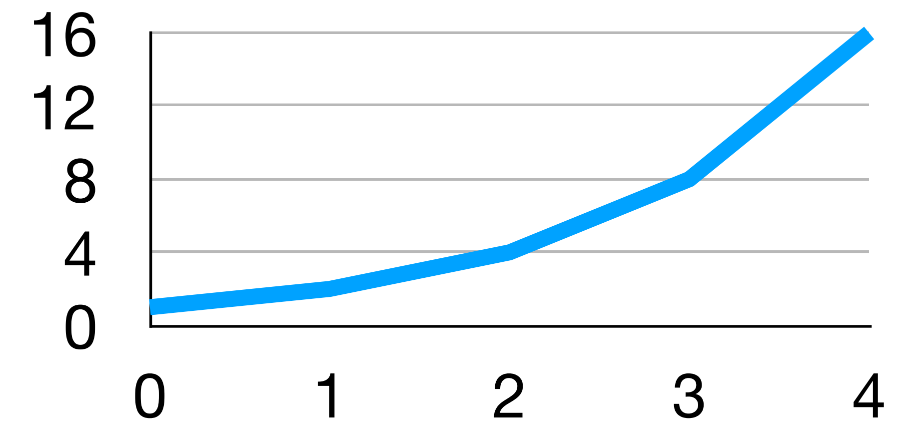
Algumas transformadas Z

Sequência geométrica: $Z\left\{a^k\right\} = \dfrac{z}{z-a}$

Esta série só converge se $|a| \leq 1$:

Se *a* = 2 teremos:

Ex.: $F(z) = \dfrac{z}{z-2}$

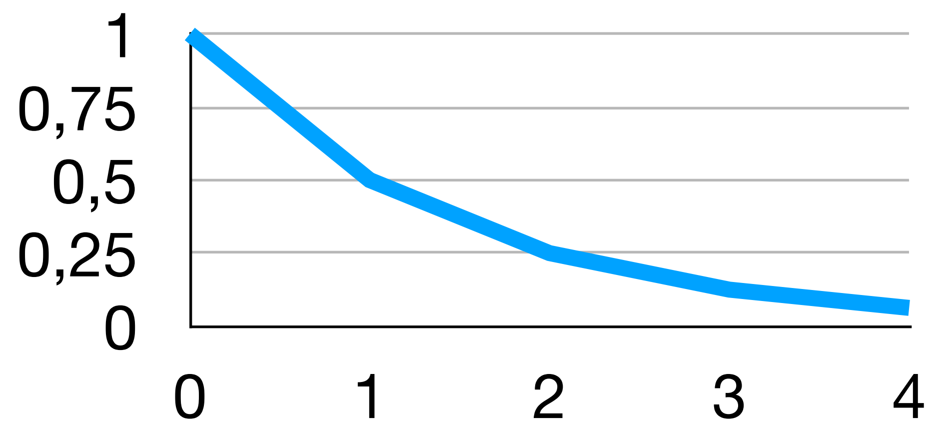


$$f[kT] = \sum_{k=0}^{\infty} (2)^k z^{-k} = 1 + 2\,z^{-1} + 4\,z^{-2} + 8\,z^{-3} + \ldots$$

Ou seja, uma função crescente! (NÃO converge)

Se *a* = 0,5 teremos:

Ex.: $F(z) = \dfrac{z}{z-0{,}5}$

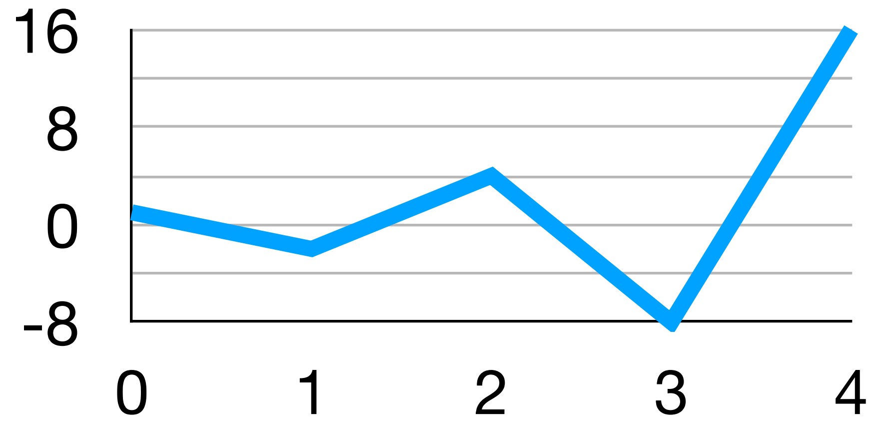


$$f[kT] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^{-k} = 1 + \frac{1}{2}\,z^{-1} + \frac{1}{4}\,z^{-2} + \frac{1}{8}\,z^{-3} + \ldots$$

Ou seja, uma função decrescente! (converge)

Se *a* = -2 teremos:

Ex.: $F(z) = \dfrac{z}{z-(-2)} = \dfrac{z}{z+2}$

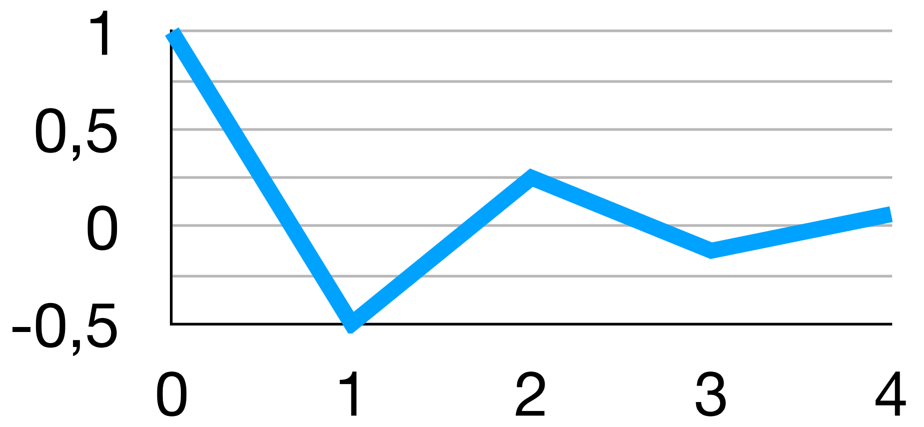


$$f[kT] = \sum_{k=0}^{\infty} (-2)^k z^{-k} = 1 - 2\,z^{-1} + 4\,z^{-2} - 8\,z^{-3} + \ldots$$

Ou seja, uma função crescente! (NÃO converge)

Se *a* = -0,5 teremos:

Ex.: $F(z) = \dfrac{z}{z-(-0{,}5)} = \dfrac{z}{z+0{,}5}$

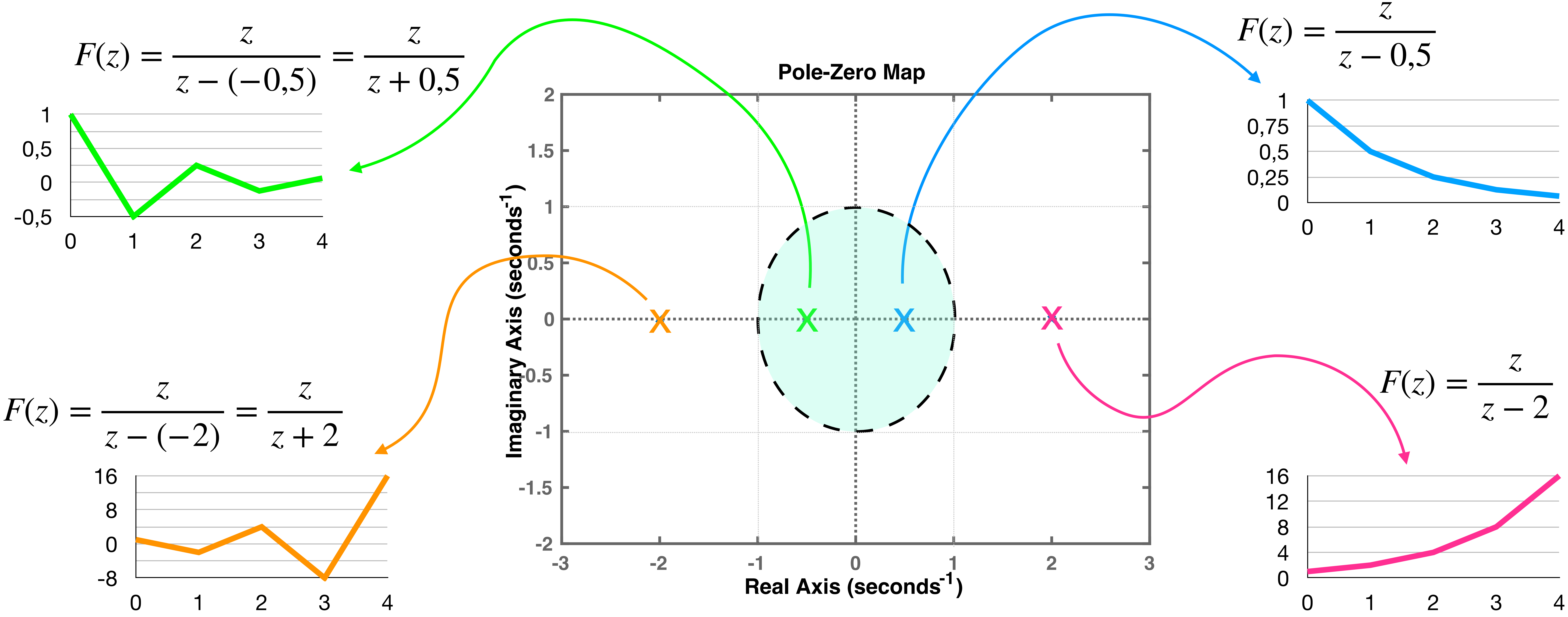


$$f[kT] = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k z^{-k} = 1 - \frac{1}{2}\,z^{-1} + \frac{1}{4}\,z^{-2} - \frac{1}{8}\,z^{-3} + \ldots$$

Ou seja, uma função decrescente! (converge)

Algumas transformadas Z

Sequência geométrica: $Z\left\{a^k\right\} = \dfrac{z}{z-a}$ ← **Esta série só converge se $|a| \leq 1$:**



$$F(z) = \frac{z}{z-(-0{,}5)} = \frac{z}{z+0{,}5}$$

$$F(z) = \frac{z}{z-0{,}5}$$

$$F(z) = \frac{z}{z-(-2)} = \frac{z}{z+2}$$

$$F(z) = \frac{z}{z-2}$$

Algumas transformadas Z

Função exponencial: $\mathcal{Z}\left\{e^{-a\,t}\right\} = \mathcal{Z}\left\{e^{-a(kT)}\right\} = \sum_{k=0}^{\infty}\left(e^{-aT}\cdot z^{-1}\right)^{k} = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + \ldots + \quad ?$

$$F(z) = \lim_{n\to\infty} \frac{1 - (\overbrace{e^{-aT}z^{-1}}^{=0})^{n+1}}{1 - e^{-aT}z^{-1}} \qquad \therefore \left[\longleftarrow \quad \sum_{k=0}^{n} aq^{k} = S(q) = \frac{a - aq^{n+1}}{1-q}\right]$$

$$F(z) = \frac{z}{z - e^{-aT}}$$

Repare:

$$\mathscr{L}\left\{e^{-at}\right\} = \frac{1}{s+a}$$

Se tenho sistema com um pólo (estável) em $s = -0{,}5$ e amostro a $T = 1{,}0$ segundo, temos:

Pela definição acima:

$$F(z) = \frac{z}{z - \left(e^{-0{,}5\cdot 1}\right)} = \frac{z}{z - 0{,}6065}$$

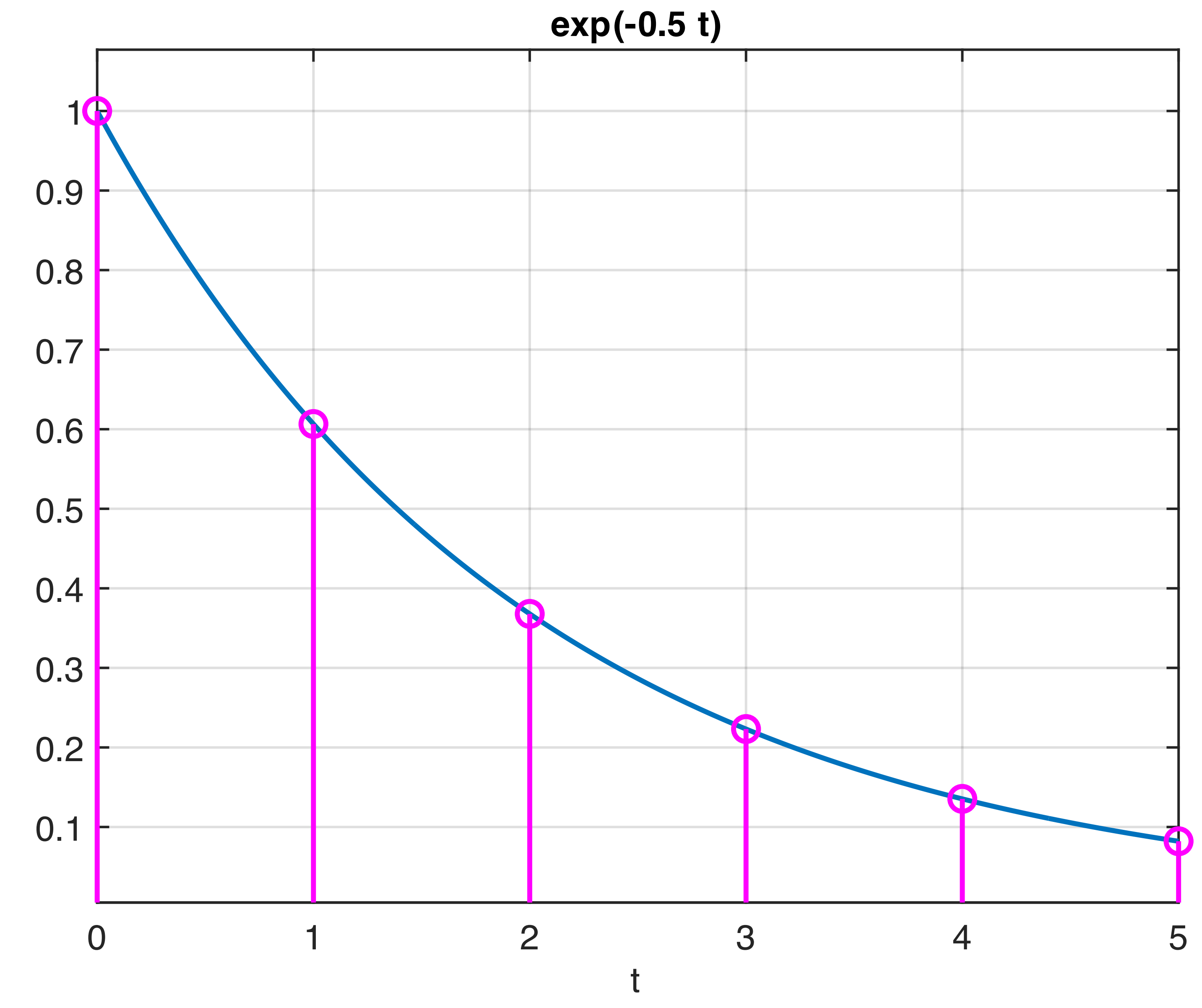
E sua inversa fica:

$$f[kT] = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + e^{-3aT}z^{-3} + \ldots$$

$$f[kT] = 1 + \left(e^{-0{,}5}\right)z^{-1} + \left(e^{-1}\right)z^{-2} + \left(e^{-1{,}5}\right)z^{-3} + \ldots$$

$$f[kT] = 1 + 0.6065\,z^{-1} + 0.3679\,z^{-2} + 0.2231\,z^{-3} + \ldots$$

Resposta ao Impulso (sistema 1ª-ordem)



Algumas transformadas Z

Função exponencial: $\mathcal{Z}\left\{e^{-a\,t}\right\} = \mathcal{Z}\left\{e^{-a(kT)}\right\} =$ $^{2aT} z^{-2} + \ldots + \quad ?$

$$F(z) = \lim_{n\to\infty} \frac{1 - (\overbrace{e^{-aT}}^{=0}}{1 - e^{-a}} \qquad) = \frac{a - aq^{n+1}}{1-q}\Bigg]$$

No Matlab:

```
>> ezplot('exp(-0.5*t)',[0 5])
>> hold on;
>> t=0:5;
>> y=exp(-0.5.*t);
>> [t' y']
ans =
         0    1.0000
    1.0000    0.6065
    2.0000    0.3679
    3.0000    0.2231
    4.0000    0.1353
    5.0000    0.0821
>> stem(t,y)
```

$$F(z) = \frac{z}{z - e^{-aT}}$$

Repare:

$$\mathscr{L}\left\{e^{-at}\right\} = \frac{1}{s+a}$$

Se tenho sistema

amostro a $T = 1$

Pela definição a

$F(z) = \dfrac{z}{z - (e^{-}}$

E sua inversa fic

$f[kT] = 1 + e^{-}$

$f[kT] = 1 + \left(e^{-0{,}5}\right) z^{-1} + \left(e^{-1}\right) z^{-2} + \left(e^{-1{,}5}\right) z^{-3} + \ldots$

$f[kT] = 1 + 0.6065\, z^{-1} + 0.3679\, z^{-2} + 0.2231\, z^{-3} + \ldots$

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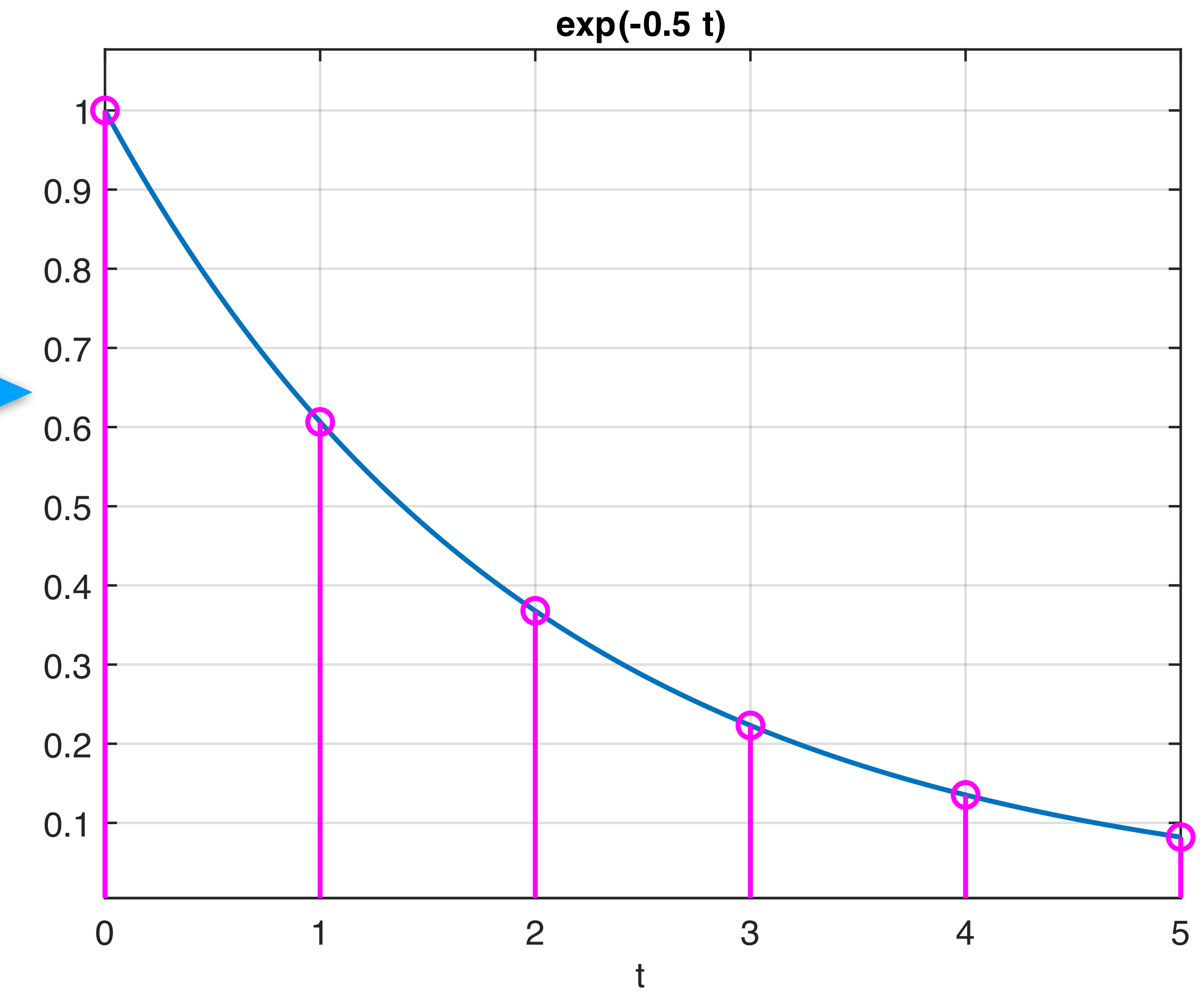

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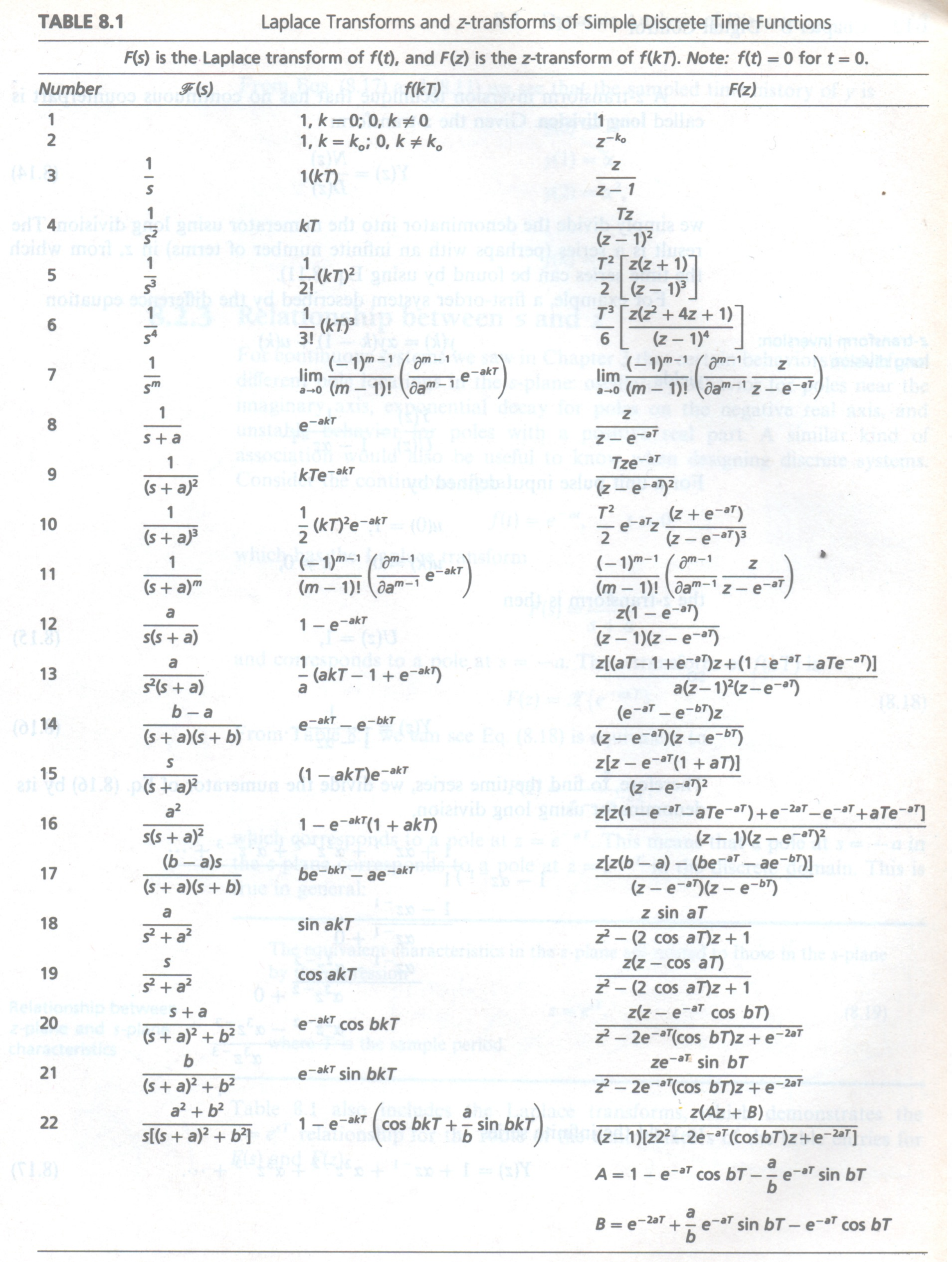
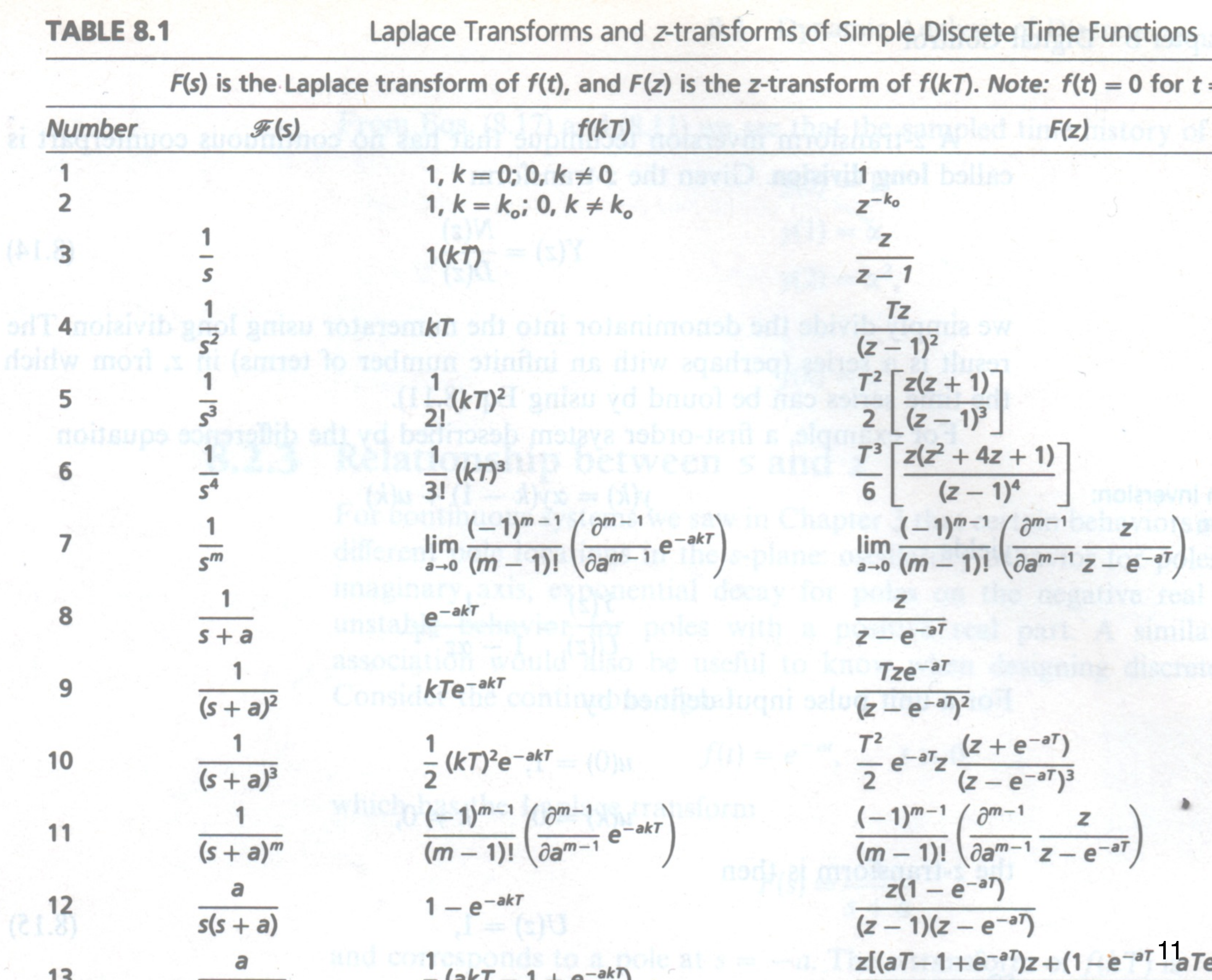


TABLE 8.1 Laplace Transforms and z-transforms of Simple Discrete Time Functions

$F(s)$ is the Laplace transform of $f(t)$, and $F(z)$ is the z-transform of $f(kT)$. *Note:* $f(t) = 0$ for $t = 0$.

Number	$\mathscr{F}(s)$	$f(kT)$	$F(z)$
1		$1, k = 0; 0, k \neq 0$	1
2		$1, k = k_o; 0, k \neq k_o$	z^{-k_o}
3	$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{1}{2!}(kT)^2$	$\frac{T^2}{2}\left[\frac{z(z+1)}{(z-1)^3}\right]$
6	$\frac{1}{s^4}$	$\frac{1}{3!}(kT)^3$	$\frac{T^3}{6}\left[\frac{z(z^2+4z+1)}{(z-1)^4}\right]$
7	$\frac{1}{s^m}$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}\right)$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}\right)$
8	$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
9	$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
10	$\frac{1}{(s+a)^3}$	$\frac{1}{2}(kT)^2e^{-akT}$	$\frac{T^2}{2}e^{-aT}z\frac{(z+e^{-aT})}{(z-e^{-aT})^3}$
11	$\frac{1}{(s+a)^m}$	$\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}\right)$	$\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}\right)$
12	$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
13	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(akT-1+e^{-akT})$	$\frac{z[(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})]}{a(z-1)^2(z-e^{-aT})}$
14	$\frac{b-a}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
15	$\frac{s}{(s+a)^2}$	$(1-akT)e^{-akT}$	$\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$
16	$\frac{a^2}{s(s+a)^2}$	$1-e^{-akT}(1+akT)$	$\frac{z[z(1-e^{-aT}-aTe^{-aT})+e^{-2aT}-e^{-aT}+aTe^{-aT}]}{(z-1)(z-e^{-aT})^2}$
17	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bkT}-ae^{-akT}$	$\frac{z[z(b-a)-(be^{-aT}-ae^{-bT})]}{(z-e^{-aT})(z-e^{-bT})}$
18	$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z \sin aT}{z^2-(2\cos aT)z+1}$
19	$\frac{s}{s^2+a^2}$	$\cos akT$	$\frac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$
20	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-akT}\cos bkT$	$\frac{z(z-e^{-aT}\cos bT)}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$
21	$\frac{b}{(s+a)^2+b^2}$	$e^{-akT}\sin bkT$	$\frac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$
22	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1-e^{-akT}\left(\cos bkT+\frac{a}{b}\sin bkT\right)$	$\frac{z(Az+B)}{(z-1)[z^2-2e^{-aT}(\cos bT)z+e^{-2aT}]}$ $A = 1-e^{-aT}\cos bT-\frac{a}{b}e^{-aT}\sin bT$ $B = e^{-2aT}+\frac{a}{b}e^{-aT}\sin bT-e^{-aT}\cos bT$



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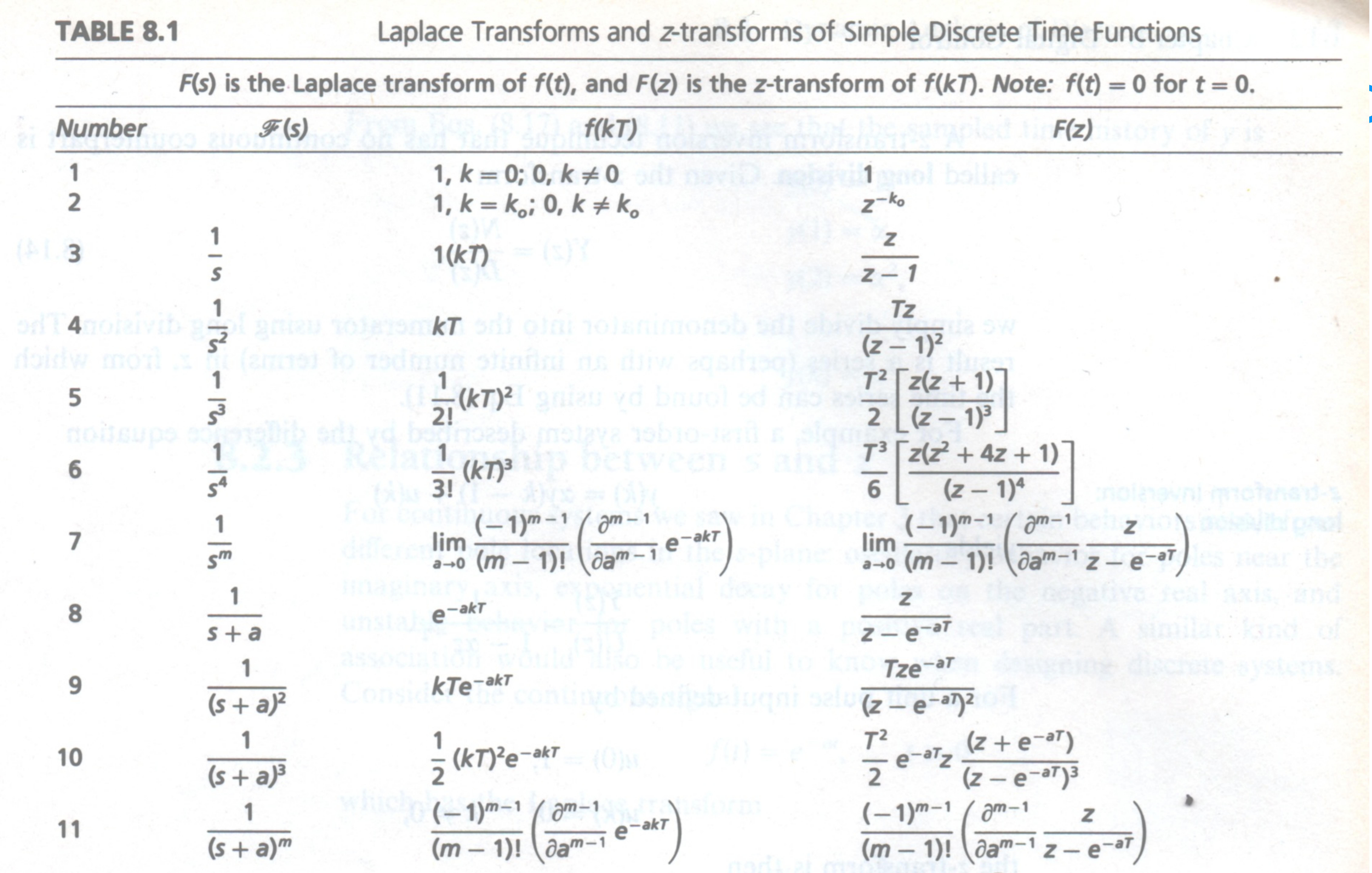

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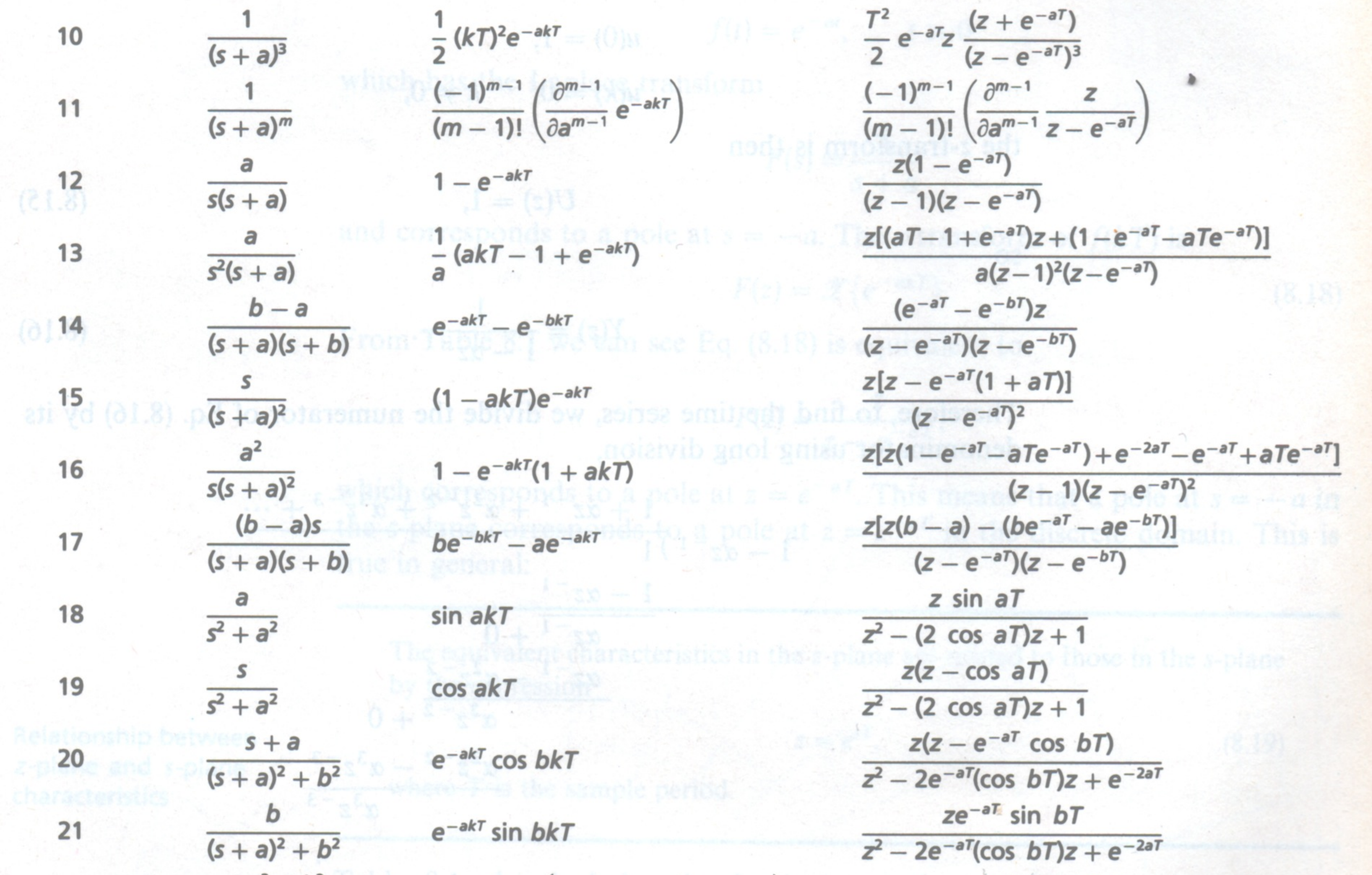


10	$\dfrac{1}{(s+a)^3}$	$\dfrac{1}{2}(kT)^2e^{-akT}$	$\dfrac{T^2}{2}e^{-aT}z\dfrac{(z+e^{-aT})}{(z-e^{-aT})^3}$
11	$\dfrac{1}{(s+a)^m}$	$\dfrac{(-1)^{m-1}}{(m-1)!}\left(\dfrac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}\right)$	$\dfrac{(-1)^{m-1}}{(m-1)!}\left(\dfrac{\partial^{m-1}}{\partial a^{m-1}}\dfrac{z}{z-e^{-aT}}\right)$
12	$\dfrac{a}{s(s+a)}$	$1-e^{-akT}$	$\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
13	$\dfrac{a}{s^2(s+a)}$	$\dfrac{1}{a}(akT-1+e^{-akT})$	$\dfrac{z[(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})]}{a(z-1)^2(z-e^{-aT})}$
14	$\dfrac{b-a}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\dfrac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
15	$\dfrac{s}{(s+a)^2}$	$(1-akT)e^{-akT}$	$\dfrac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$
16	$\dfrac{a^2}{s(s+a)^2}$	$1-e^{-akT}(1+akT)$	$\dfrac{z[z(1-e^{-aT}-aTe^{-aT})+e^{-2aT}-e^{-aT}+aTe^{-aT}]}{(z-1)(z-e^{-aT})^2}$
17	$\dfrac{(b-a)s}{(s+a)(s+b)}$	$be^{-bkT}-ae^{-akT}$	$\dfrac{z[z(b-a)-(be^{-aT}-ae^{-bT})]}{(z-e^{-aT})(z-e^{-bT})}$
18	$\dfrac{a}{s^2+a^2}$	$\sin akT$	$\dfrac{z\sin aT}{z^2-(2\cos aT)z+1}$
19	$\dfrac{s}{s^2+a^2}$	$\cos akT$	$\dfrac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$
20	$\dfrac{s+a}{(s+a)^2+b^2}$	$e^{-akT}\cos bkT$	$\dfrac{z(z-e^{-aT}\cos bT)}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$
21	$\dfrac{b}{(s+a)^2+b^2}$	$e^{-akT}\sin bkT$	$\dfrac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$

Avançar para
Propriedades da Transformada Z

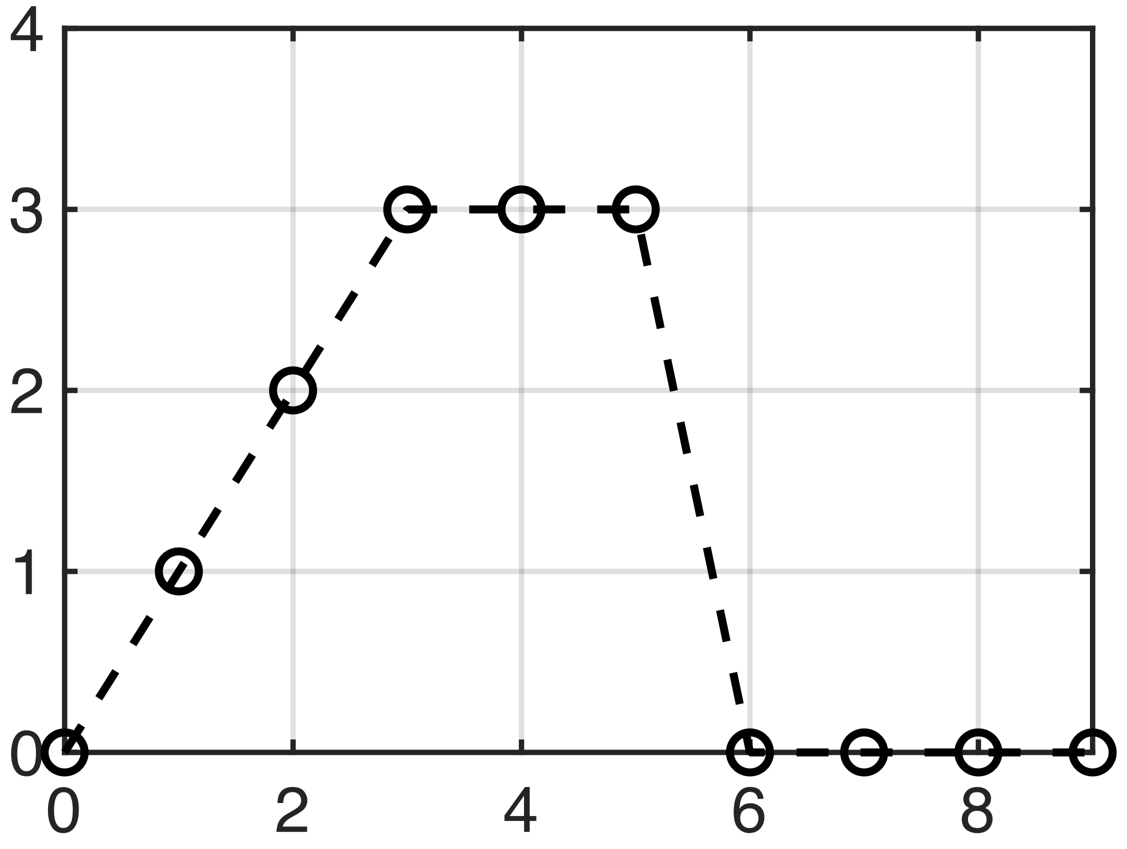
Arquivo: transformada_Z_parte2.pdf

Principais Propriedades Transformada Z

1. Atraso no tempo: $Z\{x(t - nT)\} = z^{-n}X(z)$ onde: *n* amostras de atraso (*n* = número inteiro > 0)

Seja x(t) = gráfico abaixo:

Se h(t) = x(t-2) então teremos:

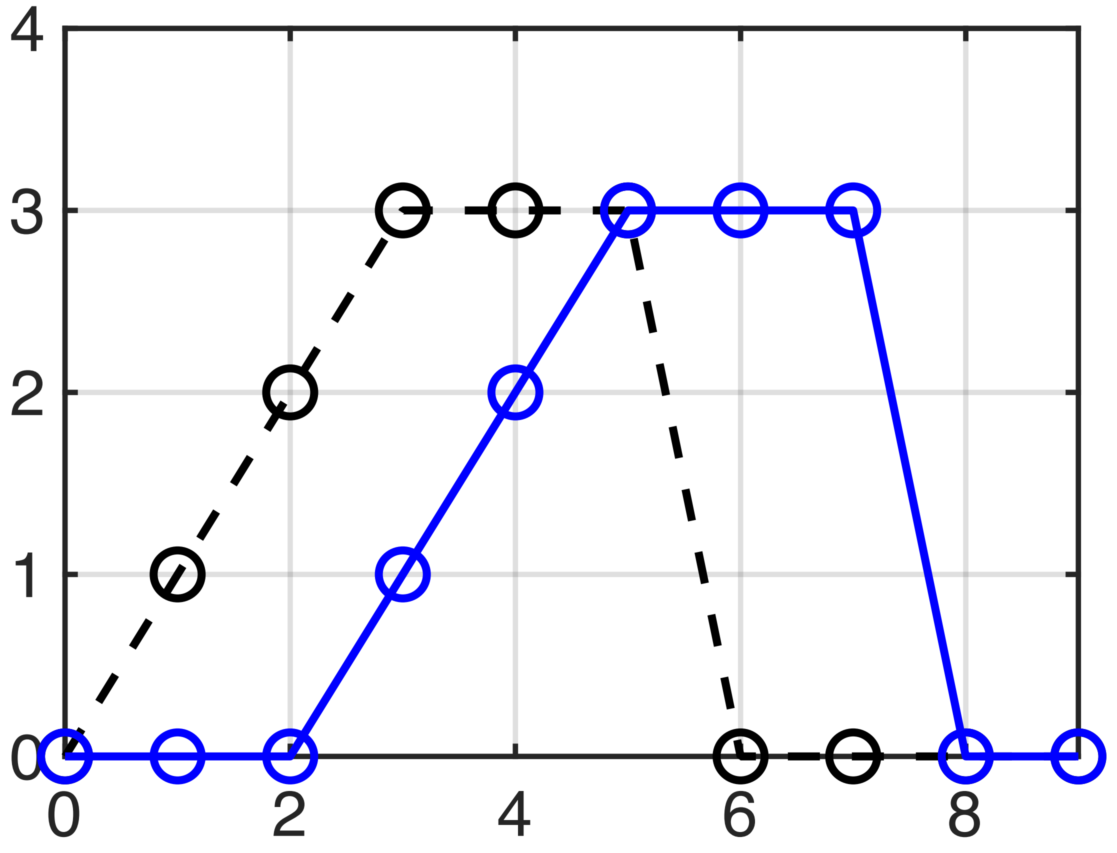


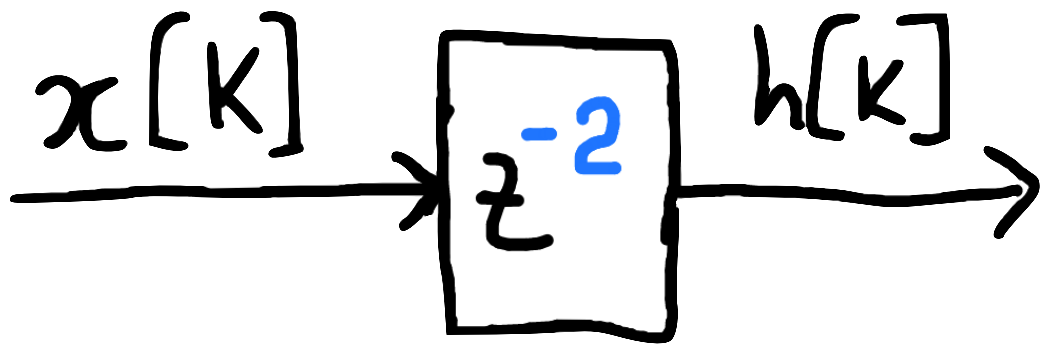
Isto é:

```
x(0)=0
x(1)=1
x(2)=2
x(3)=3
x(4)=3
x(5)=3
x(6)=0
```

Então:

```
h(0)=x(0-2)=x(-2)=0;
h(1)=x(1-2)=x(-1)=0;
h(2)=x(2-2)=x(0)=1;
h(3)=x(3-2)=x(1)=1;
h(4)=x(4-2)=x(2)=2;
h(5)=x(5-2)=x(3)=3;
h(6)=x(6-2)=x(4)=3;
h(7)=x(7-2)=x(5)=3;
h(8)=x(8-2)=x(6)=0;
h(9)=x(9-2)=x(7)=0;
```





Principais Propriedades Transformada Z

2. Teorema do Valor final: $f(\infty) = \lim_{k \to \infty} f(k) = \lim_{z \to 1} (1 - z^{-1})\, F(z)$

$$Y(z) = \frac{z}{(z-1)(z-a)}$$

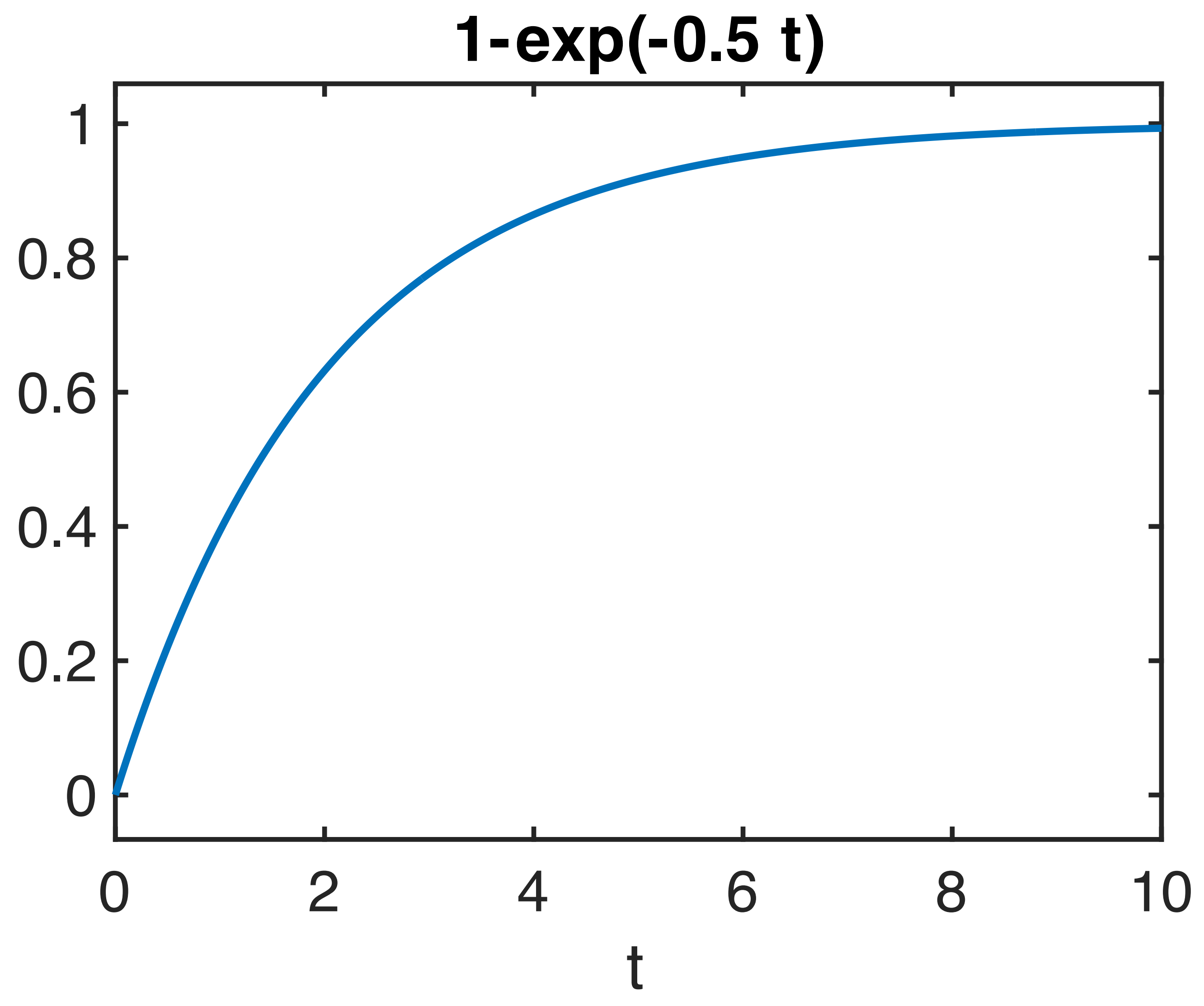
$$y(\infty) = \lim_{z \to 1} (1 - z^{-1})\, Y(z) = \lim_{z \to 1} \left(\frac{z-1}{z} \right) Y(z) = \lim_{z \to 1} \frac{1}{(z-a)} = \frac{1}{1-a}$$

Principais Propriedades Transformada Z

2. Teorema do Valor final: $f(\infty) = \lim_{k\to\infty} f(k) = \lim_{z\to 1}(1 - z^{-1})\,F(z)$

$$x(\infty) = ? \qquad \text{de} \qquad X(z) = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-aT} z^{-1}} \qquad (a > 0)$$

$$\begin{aligned} x(\infty) &= \lim_{z\to 1} \left[(1 - z^{-1})\,X(z)\right] \\ &= \lim_{z\to 1} \left[(1 - z^{-1}) \left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-aT} z^{-1}}\right)\right] \\ &= \lim_{z\to 1} \left(1 - \frac{1 - z^{-1}}{1 - e^{-aT} z^{-1}}\right) = 1 \end{aligned}$$



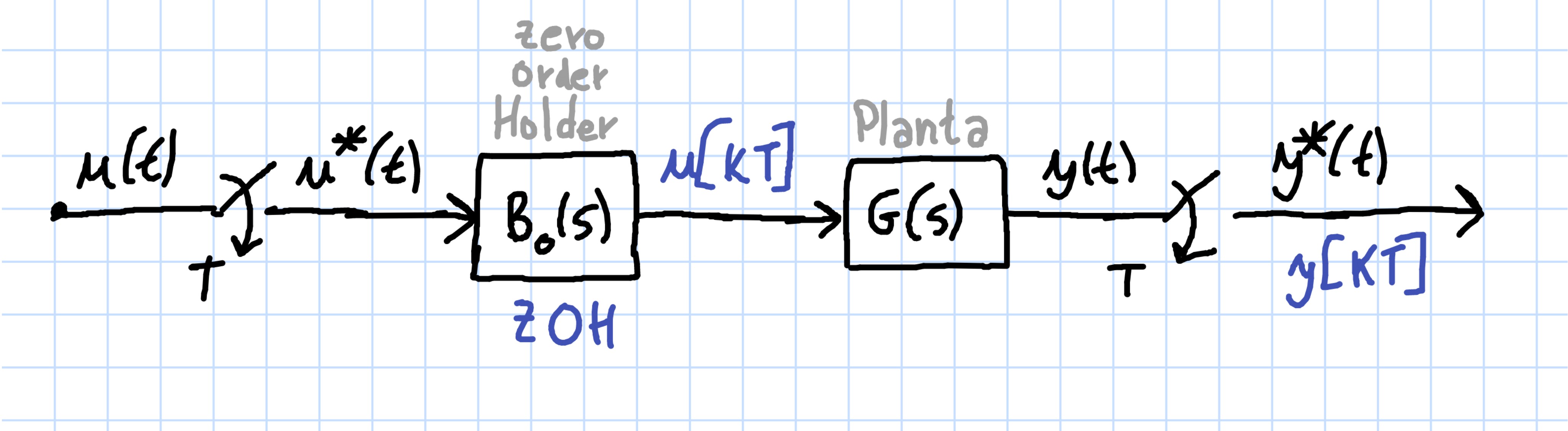
Note que $X(z)$ é a transformada Z de $x(t) = 1 - e^{-at}$.

$$x(\infty) = \lim_{t\to\infty} (1 - e^{-at}) = 1$$

Avançar para BoG(z)

Arquivo: 3_BoG_Transformada_Z.pdf

BoG(z)



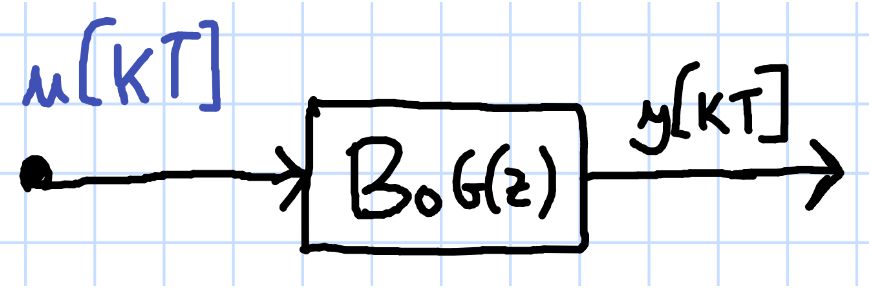
$$Bo(s) = \frac{1 - e^{-Ts}}{s}$$

$$\frac{Y(s)}{U(s)} = \left(\frac{1 - e^{-Ts}}{s} \right) G(s)$$

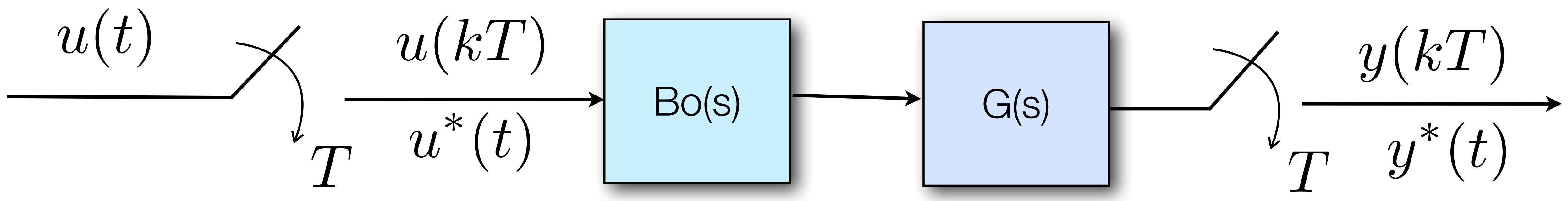
Atraso de 1 amostra

$$Z\{Bo(s) \cdot G(s)\} = Z\left\{ \left(\frac{1 - e^{-Ts}}{s} \right) \cdot G(s) \right\} = Z\left\{ \frac{G(s)}{s} \right\} - Z\left\{ \frac{e^{-Ts} G(s)}{s} \right\}$$

$$BoG(z) = Z\{Bo(s)G(s)\} = \left(1 - z^{-1}\right) \cdot Z\left\{ \frac{G(s)}{s} \right\} = \frac{z-1}{z} \cdot Z\left\{ \frac{G(s)}{s} \right\}$$



BoG(z): Exemplo c/sistema 1ª-ordem:



Se $G(s) = \dfrac{K\cdot g}{(1+\tau s)}$ (Um sistema de 1a-ordem)

Aplicando o que acabamos de verificar: $BoG(z) = (1-z^{-1})\cdot \mathcal{Z}\left\{\dfrac{G(s)}{s}\right\}$, obtemos:

$$BoG(z) = \mathcal{Z}\left\{\frac{K\cdot g\cdot(1-e^{-Ts})}{s(1+\tau s)}\right\} = (1-z^{-1})\cdot\mathcal{Z}\left\{\frac{K\cdot g}{s(1+\tau s)}\right\} = K\,g\,(1-z^{-1})\cdot\mathcal{Z}\left\{\frac{1}{s(1+\tau s)}\right\}$$

$$\frac{1}{s(1+\tau s)} = \frac{A}{s}+\frac{B}{(1+\tau s)} = \frac{A(1+\tau s)+Bs}{s(1+\tau s)} = \frac{A+s(B+\tau A)}{s(1+\tau s}$$

$$A = \left.\frac{1}{(s+\tau s)}\right|_{s=0} = 1$$

$$B = \left.\frac{1}{s}\right|_{s=1/(-\tau)} = -\tau$$

$$\frac{1}{s(1+\tau s)} = \frac{1}{s} - \frac{\tau}{(1+\tau s)} = \frac{1}{s} - \frac{1}{1/\tau + s}$$

$$\mathcal{Z}\left\{\frac{1}{s(s+\tau s)}\right\} = \frac{z}{z-1} - \frac{z}{z-e^{-T/\tau}} \quad \text{ou}$$

$$\mathcal{Z}\left\{\frac{1}{s(s+\tau s)}\right\} = \frac{1}{1-z^{-1}} - \frac{1}{1-z^{-1}\cdot e^{-T/\tau}}$$

$$BoG(z) = \frac{K\,g\,(1-z^{-1})\cdot 1}{(1-z^{-1})} - \frac{K\,g\,(1-z^{-1})\cdot 1}{(1-z^{-1}\cdot e^{-T/\tau})}$$

$$BoG(z) = Kg - \frac{Kg(z-1)}{(z-e^{-T/\tau})} = \frac{Kg(z-e^{-T/\tau})-Kg(z-1)}{(z-e^{-T/\tau})} = \frac{Kg\,z - Kg\,e^{-T/\tau} - Kg\,z + Kg}{(z-e^{-T/\tau})}$$

$$BoG(z) = \frac{K\,g\,(1-e^{-T/\tau})}{(z-e^{-T/\tau})}$$

Sistema de 1ª-ordem (plano-s x plano-z)

Seja: $G(s)=\dfrac{2}{s+2}$

Se for aplicado um degrau a sua entrada teremos:

$$Y(s)=U(s)\cdot G(s)$$

$$Y(s)=\frac{1}{s}\cdot\frac{2}{(s+2)}=2\cdot\left[\frac{R_1}{s}+\frac{R_2}{s+2}\right]$$

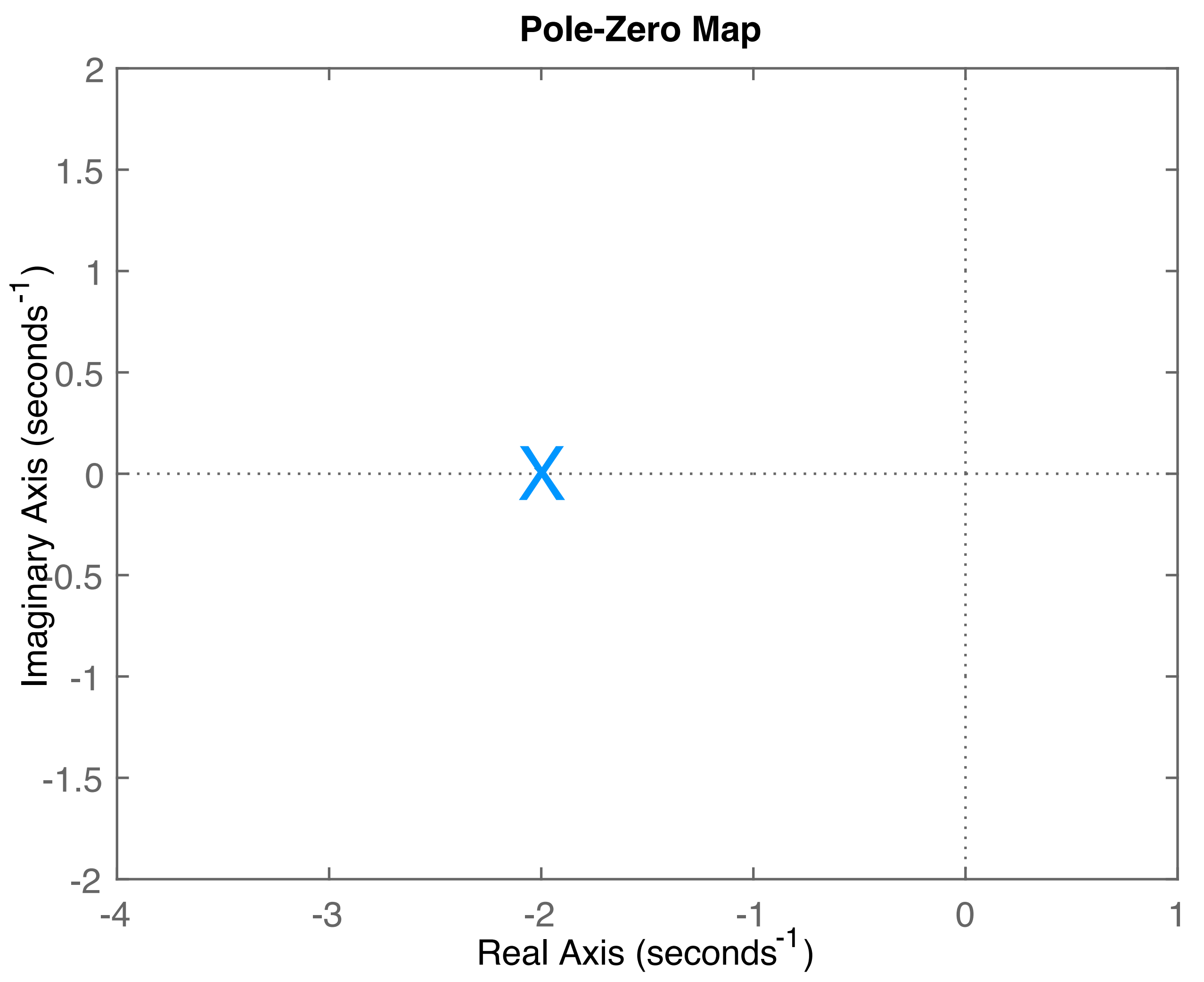
$$R_1=\left.\frac{s\cdot 1}{s(s+2)}\right|_{s=0}=\frac{1}{2}$$

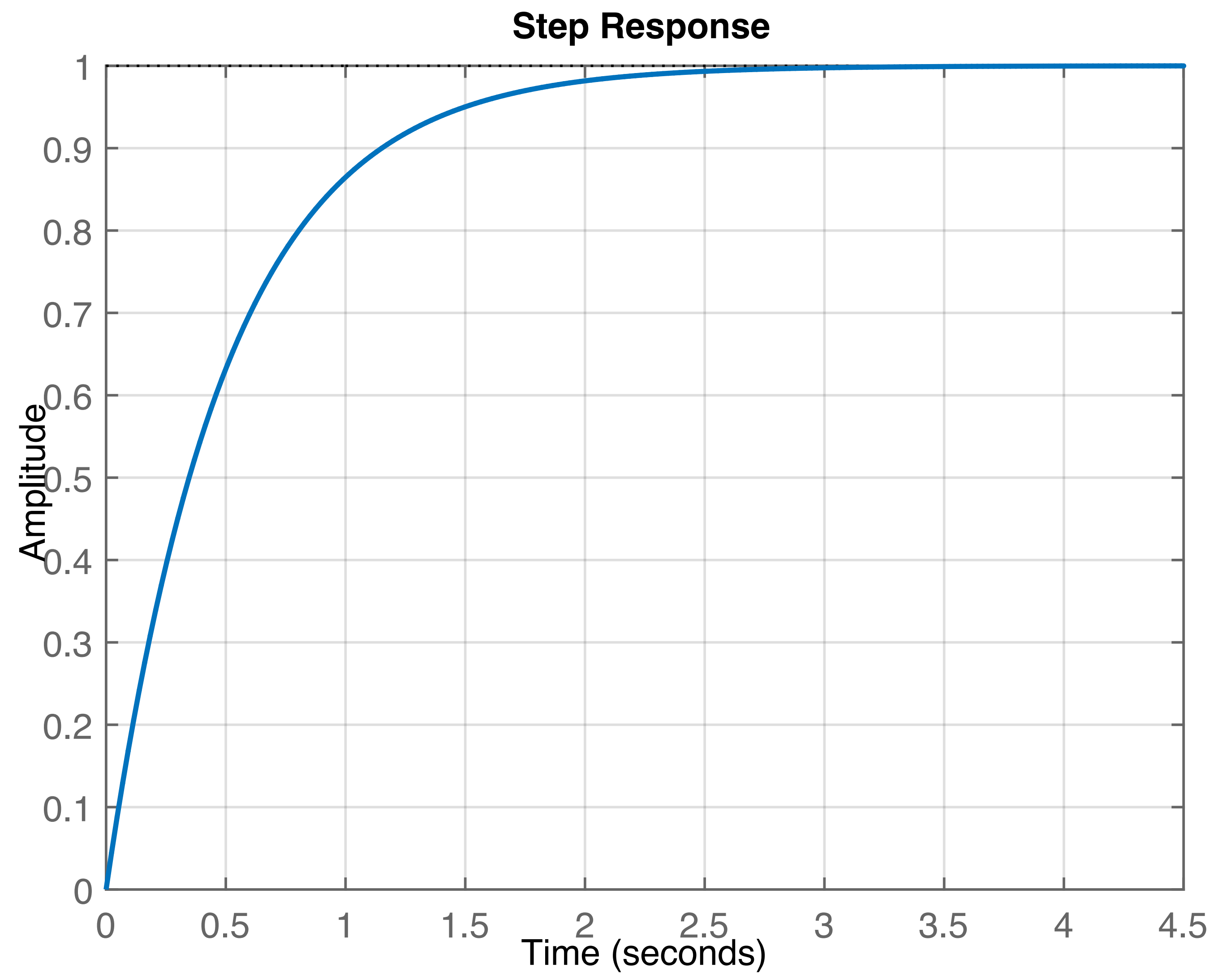
$$R_2=\left.\frac{(s+2)\cdot 1}{s(s+2)}\right|_{s=-2}=\left.\frac{1}{s}\right|_{s=-2}=-\frac{1}{2}$$

$$y(t)=\mathscr{L}^{-1}\left\{Y(s)\right\}=\cancel{2}\cdot\frac{1}{\cancel{2}}\left(1-e^{-2t}\right)$$

No Matlab:

```
>> t=[0:0.5:4];
>> y=1*(1-exp(-2.*t));
>> [t' y’]
ans =
         0         0
    0.5000    0.6321
    1.0000    0.8647
    1.5000    0.9502
    2.0000    0.9817
    2.5000    0.9933
    3.0000    0.9975
    3.5000    0.9991
    4.0000    0.9997
>> G=tf(2,[1  2]);
>> step(G)
```





Sistema de 1ª-ordem (plano-s x plano-z)

Seja: $G(s)=\dfrac{2}{s+2}=\dfrac{A}{s+a}$

Amostrando este sistema à T=1,0 segundos, teremos:

$$BoG(z)=(1-z^{-1})\cdot\mathbb{Z}\left\{\frac{G(s)}{s}\right\}=\frac{(z-1)}{z}\cdot\mathbb{Z}\left\{\frac{G(s)}{s}\right\}$$

$$\frac{G(s)}{s}=2\cdot\left[\frac{1}{s(s+2)}\right]=2\left[\frac{R_1}{s}+\frac{R_2}{(s+2)}\right]$$

$$R_1=\frac{1}{2}$$

$$R_2=-\frac{1}{2}$$

$$BoG(z)=2\cdot\frac{(z-1)}{z}\cdot\mathbb{Z}\left\{\frac{1/2}{s}-\frac{1/2}{(s+2)}\right\}$$

$$BoG(z)=1\cdot\frac{(z-1)}{z}\cdot\left[\frac{z}{z-1}-\frac{z}{z-e^{-2}}\right]$$

$$BoG(z)=1\left[\frac{z(z-1)}{z(z-1)}-\frac{z(z-1)}{z(z-0.1353)}\right]$$

De uma tabela da transformadas Z e de Laplace:

$$\frac{1}{s}\rightleftharpoons\frac{z}{z-1}\quad\text{e}\quad\frac{1}{s+a}\rightleftharpoons\frac{z}{z-e^{-aT}}$$

$$BoG(z)=1\left[1-\frac{(z-1)}{(z-0{,}1353)}\right]$$

$$BoG(z)=1\cdot\left[\frac{1-0{,}1353}{z-0{,}1353}\right]=\frac{0{,}8647}{z-0{,}1353}$$

Ou (mais genericamente):

$$BoG(z)=\frac{A}{a}\cdot\frac{(1-e^{-aT})}{(z-e^{-aT})}$$

No Matlab:

```
>> G=tf(2,[1 2]);
>> T=1;
>> BoG=c2d(G,T);
>> zpk(BoG)

  0.86466
 ----------
 (z-0.1353)

>>
```

Sistema de 1ª-ordem (plano-s x plano-z)

Seja: $G(s) = \dfrac{2}{s+2} = \dfrac{A}{s+a}$

Amostrando este sistema à T=1,0 segundos, teremos:

$$BoG(z) = \frac{A}{a} \cdot \frac{(1-e^{-aT})}{(z-e^{-aT})} = \frac{0{,}8647}{z-0{,}1353}$$

Submetendo este sistema a uma entrada degrau (unitário), teremos:

$$Y(z) = U(z) \cdot G(z)$$

$$Y(z) = \frac{z}{(z-1)} \cdot \frac{A}{a} \cdot \frac{(1-e^{-aT})}{(z-e^{-aT})}$$

$$\frac{Y(z)}{z} = \frac{A(1-e^{-aT})}{a} \cdot \frac{1}{(z-1)(z-e^{-aT})}$$

$$\frac{Y(z)}{z} = \frac{A(1-e^{-aT})}{a} \left[\frac{R_1}{(z-1)} + \frac{R_2}{(z-e^{-aT})} \right]$$

$$R_1 = \frac{1}{(1-e^{-aT})}$$

$$R_2 = \frac{1}{e^{-aT}-1} = -\frac{1}{1-e^{-aT}}$$

$$\frac{Y(z)}{z} = \frac{A(1-e^{-aT})}{a} \left[\frac{R_1\, z}{(z-1)} + \frac{R_2\, z}{(z-e^{-aT})} \right]$$

De uma tabela de transformadas Z:

$$\frac{z}{(z-1)} \rightleftharpoons u[kT] \quad \textbf{e} \quad \frac{z}{(z-e^{-aT})} = \frac{z}{(z-p)} \rightleftharpoons p^k$$

$$y[kT] = \frac{A(1-e^{-aT})}{a} \left[R_1\, u[kT] + R_2\, (e^{-aT})^k \right]$$

$$y[kT] = \frac{A(1-e^{-aT})}{a} \cdot \left[\frac{u[kT]}{(1-e^{-aT})} - \frac{(e^{-aT})^k}{(1-e^{-aT})} \right]$$

$$y[kT] = \frac{A}{a} \left(1-p^k\right)$$

onde: $p = e^{-aT} = e^{-2 \cdot 1} = 0{,}1353$

$$y[kT] = 1 - (0{,}1353)^k \quad \forall\ k > 0$$

Comparando com: $y(t) = \mathscr{L}^{-1}\left\{Y(s)\right\} = \cancel{2} \cdot \dfrac{1}{\cancel{2}} \left(1 - e^{-2t}\right)$

Sistema de 1ª-ordem (plano-s x plano-z)

Seja: $G(s) = \dfrac{2}{s+2} = \dfrac{A}{s+a}$

Submetendo este sistema a uma entrada degrau (unitário), teremos:

$$Y(z) = U(z) \cdot G(z)$$

$$Y(z) = \frac{z}{(z-1)} \cdot \frac{A}{a} \cdot \frac{(1-e^{-aT})}{(z-e^{-aT})}$$

$$y[kT] = \frac{A}{a}\left(1-p^k\right)$$

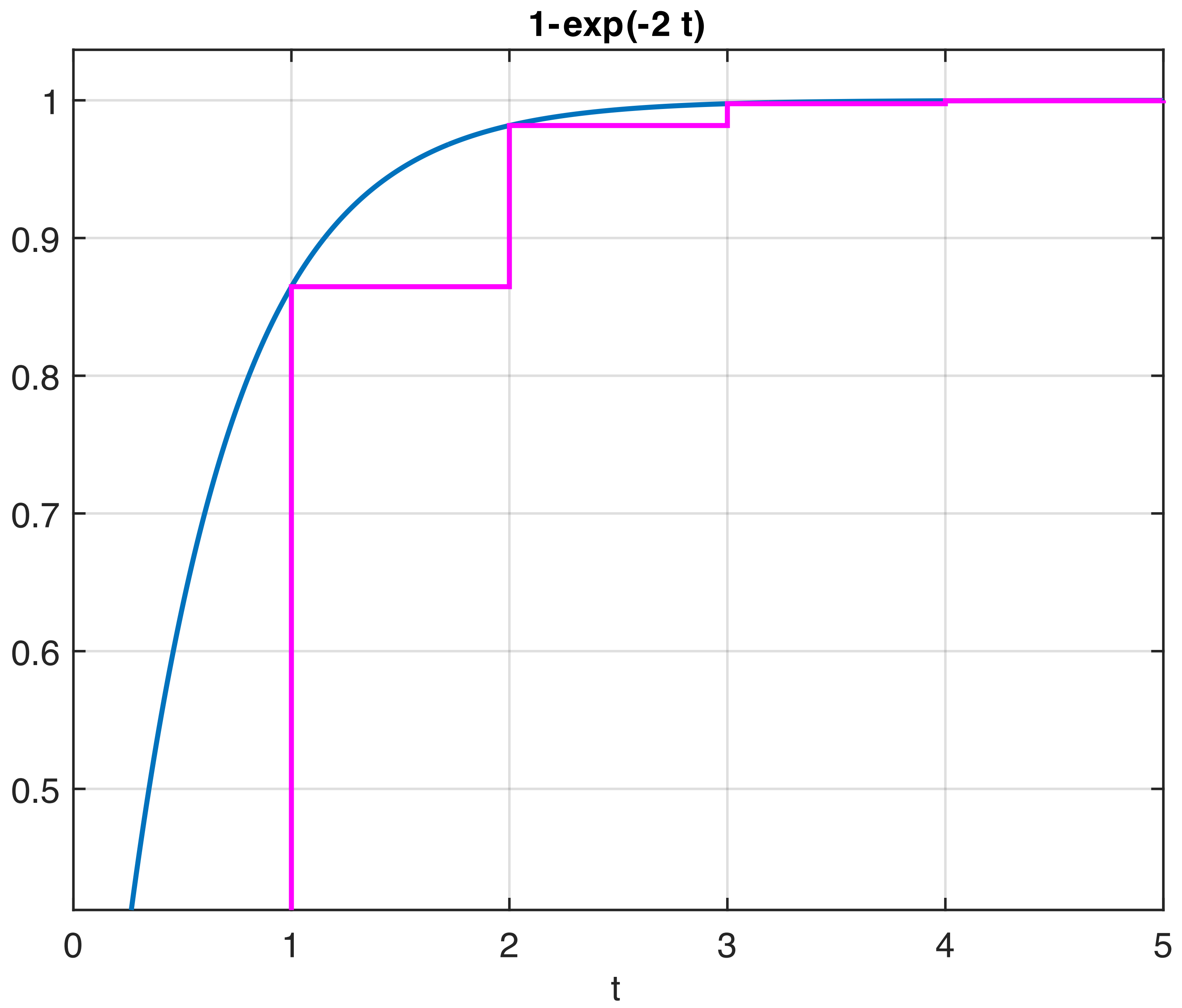
onde: $p = e^{-aT} = e^{-2\cdot 1} = 0{,}1353$

$y[kT] = 1 - (0{,}1353)^k \quad \forall\, k > 0$ (mundo discreto)

$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \cancel{2} \cdot \dfrac{1}{\cancel{2}}\left(1-e^{-2t}\right)$ (mundo contínuo)

No Matlab:

```
>> T=1.0;
>> a=2;
>> p=exp(-a*T)
p =
    0.1353
>> k=0:5; % vetor amostras
>> t=0:5; % vetor tempo contínuo
>> y_cont=1-exp(-a.*t);
>> y_disc=1-p.^k;
>> [t' y_cont' y_disc']
ans =
         0         0         0
    1.0000    0.8647    0.8647
    2.0000    0.9817    0.9817
    3.0000    0.9975    0.9975
    4.0000    0.9997    0.9997
    5.0000    1.0000    1.0000
>> figure; ezplot('1-exp(-2*t)',[0 5])
>> hold on;
>> stairs(k,y_disc) % plotando amostras digitalizadas
>> grid
```



Sistema de 1ª-ordem (plano-s x plano-z)

Seja: $G(s) = \dfrac{2}{s+2} = \dfrac{A}{s+a}$

Submetendo este sistema a uma entrada degrau (unitário), teremos:

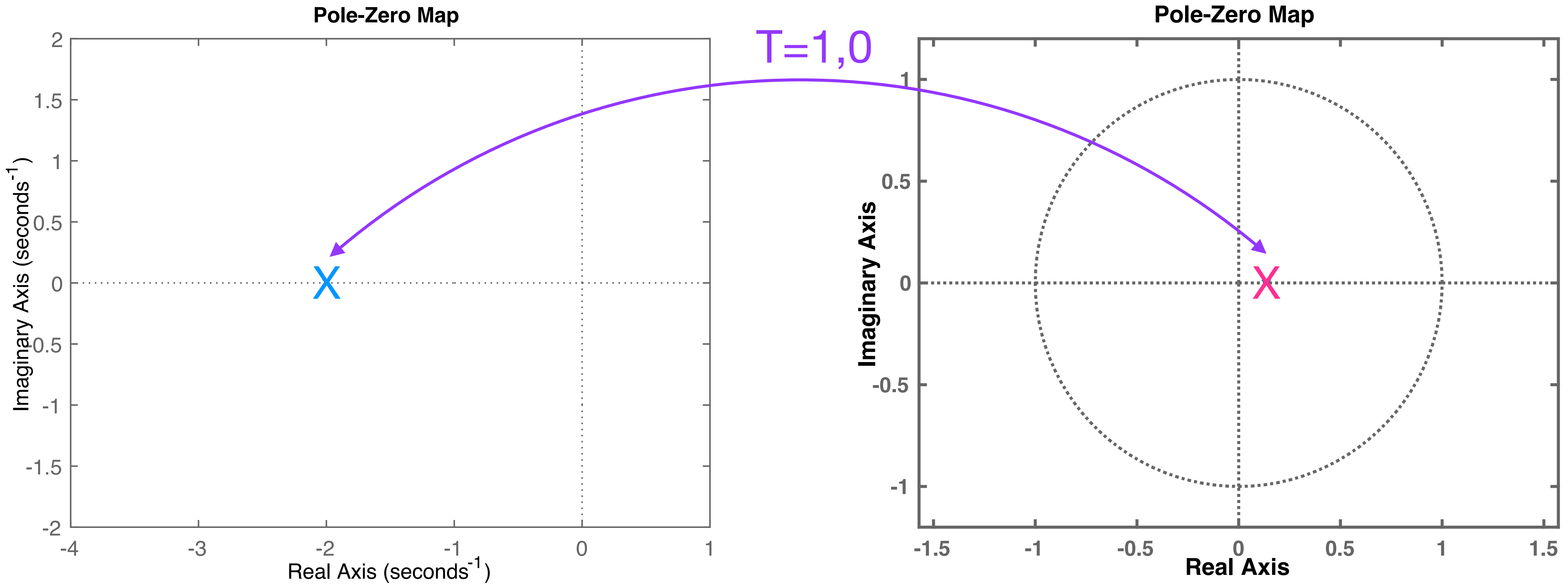
$$Y(z) = U(z) \cdot G(z)$$
$$Y(z) = \frac{z}{(z-1)} \cdot \frac{A}{a} \cdot \frac{(1-e^{-aT})}{(z-e^{-aT})}$$

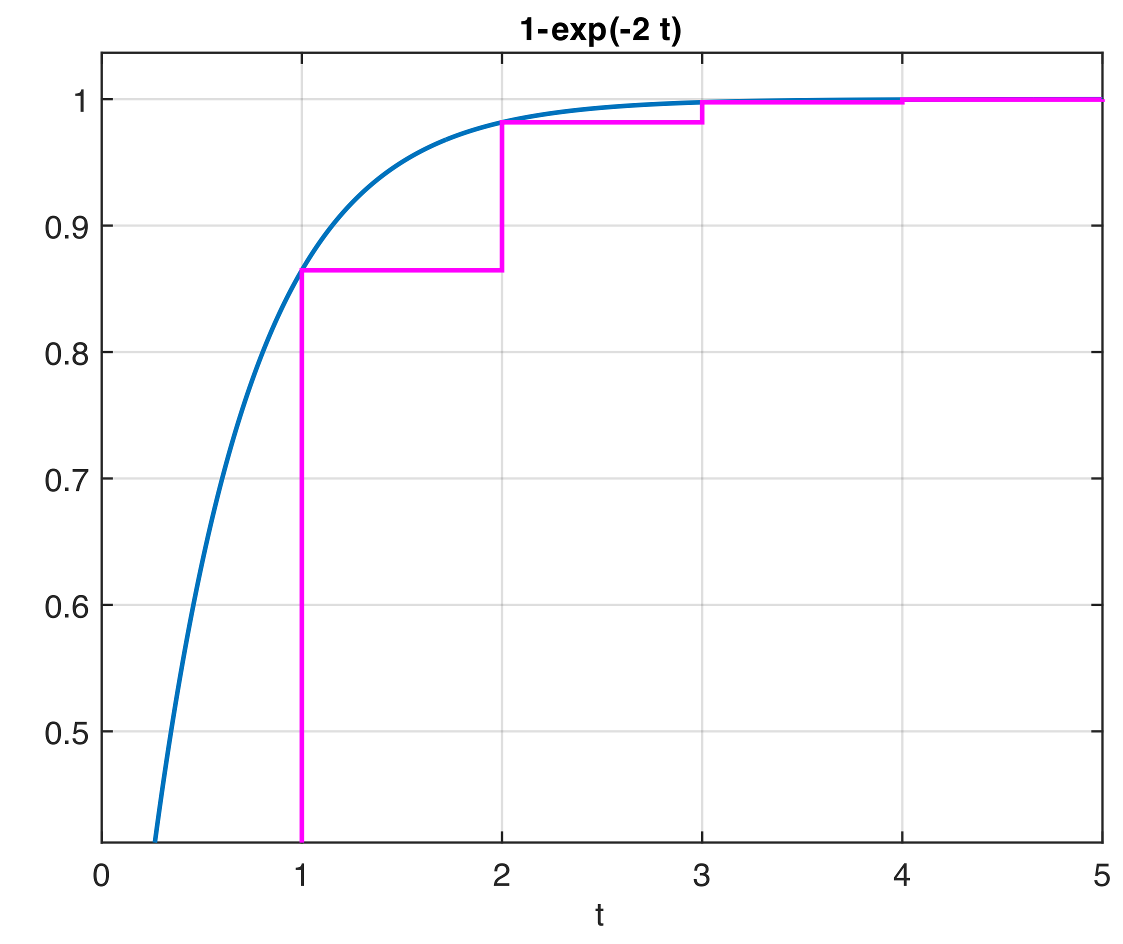
$$y[kT] = \frac{A}{a}\left(1-p^k\right)$$

onde: $p = e^{-aT} = e^{-2\cdot 1} = 0{,}1353$

$y[kT] = 1 - (0{,}1353)^k \quad \forall\ k > 0$ (mundo discreto)

$y(t) = \mathscr{L}^{-1}\left\{Y(s)\right\} = \cancel{2} \cdot \dfrac{1}{\cancel{2}}\left(1-e^{-2t}\right)$ (mundo contínuo)





Avançar para Métodos de Transformadas Inversa de Z

Arquivo: transformada_Z_parte_3.pdf