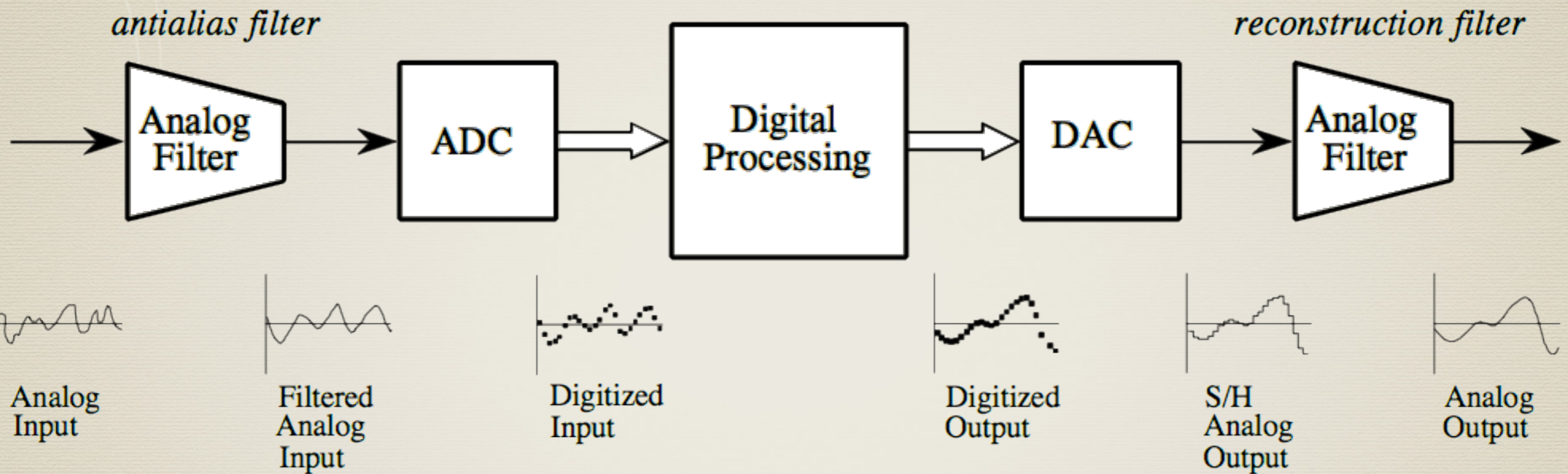


SISTEMAS AMOSTRADOS

(Teorema da Amostragem)

Prof. Fernando Passold

Digitalização de um Sinal



O Processo de Digitalização

Dados:

Sinal de entrada:
0 ~ 4.095 Volts;
A/D de 12-bits:
saída: 0 ~ 4095

Ref.: Chap 3. of The Scientist and Engineer's Guide to Digital Signal Processing, 2nd. ed. , Steven W. Smith, California Technical Publishing, 1.999
URL: <http://www.dspguide.com/pdfbook.htm>

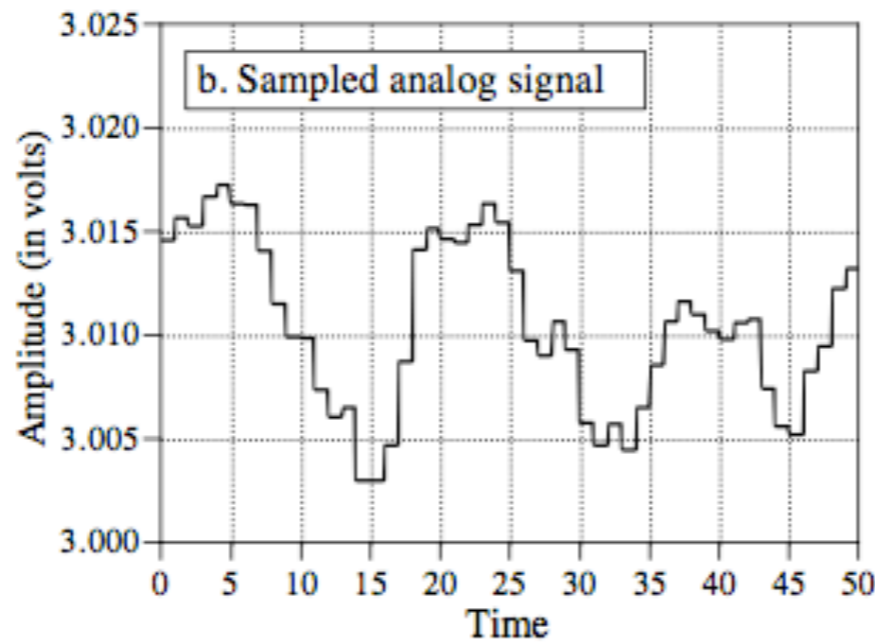
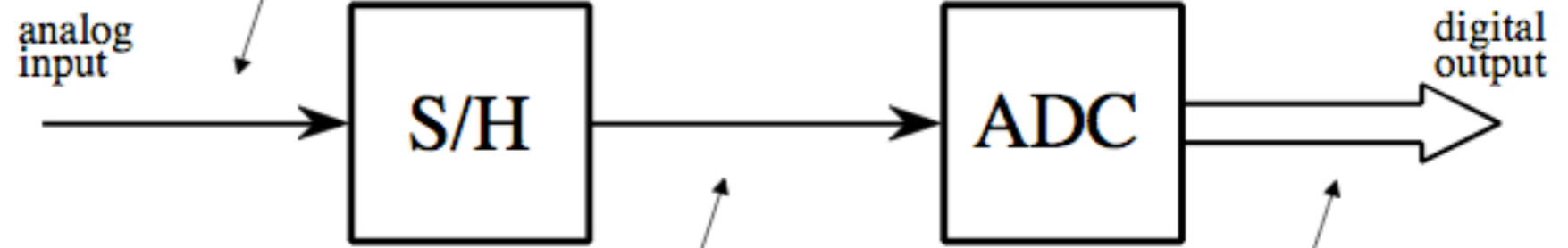
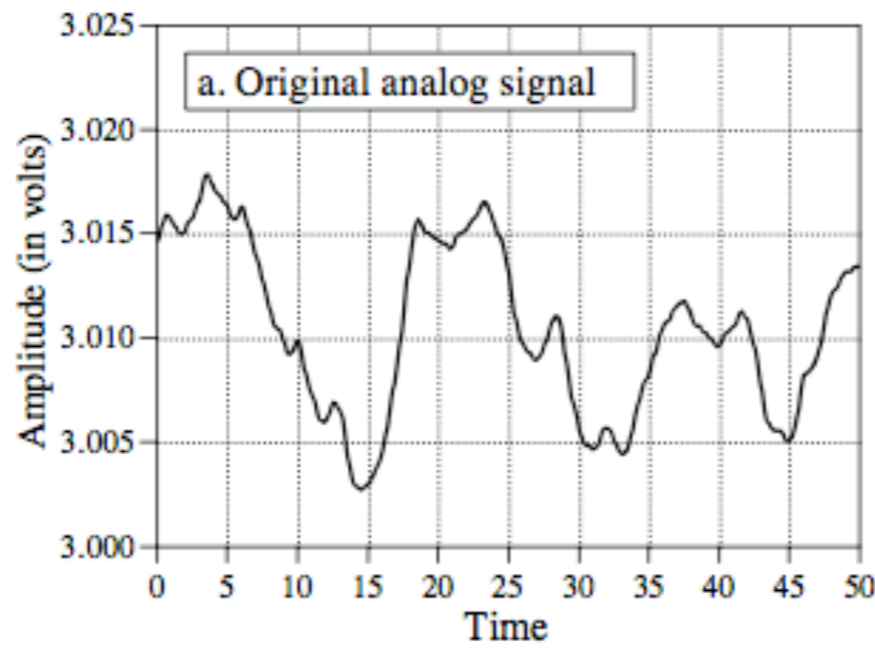
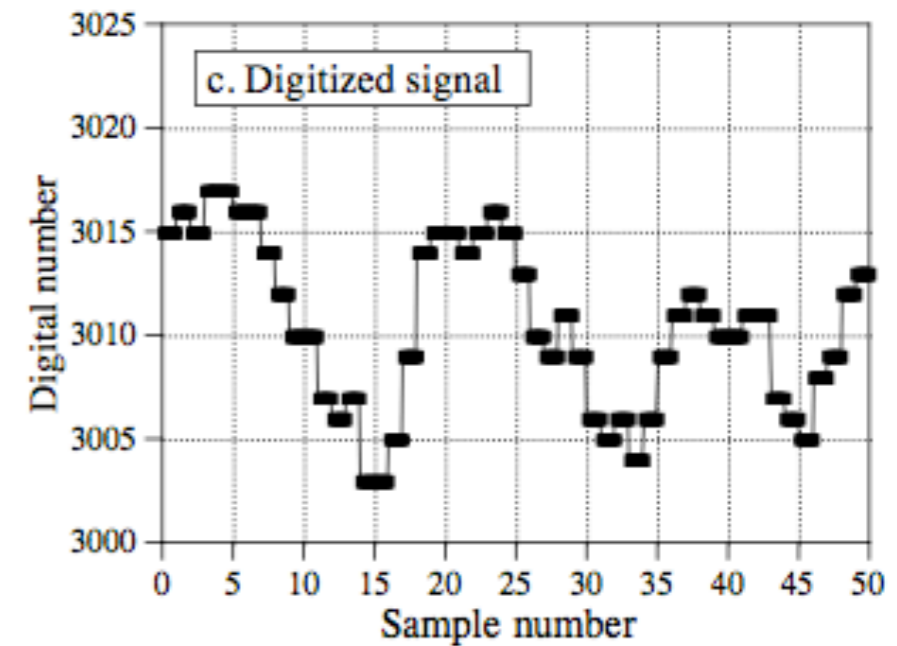


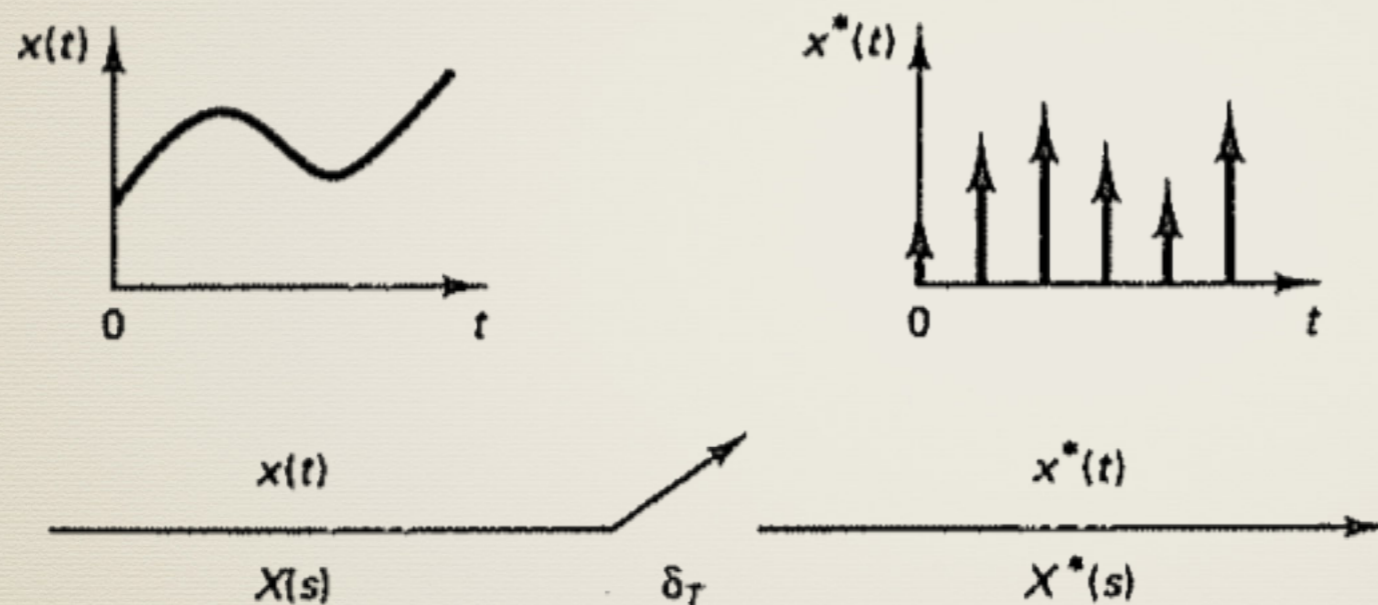
FIGURE 3-1

Waveforms illustrating the digitization process. The conversion is broken into two stages to allow the effects of *sampling* to be separated from the effects of *quantization*. The first stage is the sample-and-hold (S/H), where the only information retained is the instantaneous value of the signal when the periodic sampling takes place. In the second stage, the ADC converts the voltage to the nearest integer number. This results in each sample in the digitized signal having an error of up to $\pm\frac{1}{2}$ LSB, as shown in (d). As a result, quantization can usually be modeled as simply adding noise to the signal.



Sinal Amostrado

* Suponha que um sistema contínuo no tempo, $x(t)$ esteja sendo amostrado por um trem de impulsos deslocados no tempo:



Pág. 75, Ogata, Cap. 3: Z-plane Analysis of Discrete-Time Control Systems.
 URL: http://een.iust.ac.ir/profs/Jahed/digital%20controll/e%20book/discrete-time_control_systems.pdf

Figure 3-1 Impulse sampler.

$$(3.1) \quad x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT)$$

Trem de impulsos deslocados no tempo (múltiplos do período de amostragem)

$$x^*(t) = x(0)\delta(t) + x(T)\delta(t - T) + \dots + x(kT)\delta(t - kT) + \dots$$

Sinal Amostrado

* O processo de amostragem por trem de pulsos pode ser abordado como um sinal de entrada $x(t)$ que foi modulado por um trem de impulsos unitários, $\delta(t)$

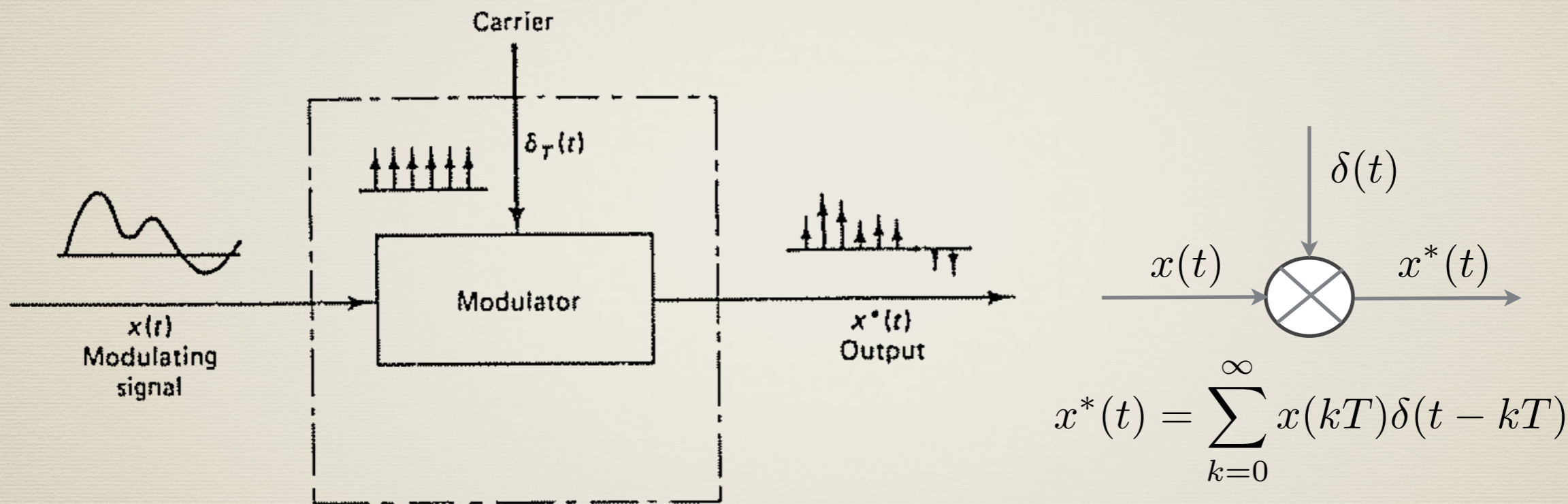


Figure 3-2 Impulse sampler as a modulator.

$$X^*(s) = \mathcal{L}[x^*(t)] = x(0)\mathcal{L}[\delta(t)] + x(T)\mathcal{L}[\delta(t - T)] + x(2T)\mathcal{L}[\delta(t - 2T)] + \dots$$

- Realizando a transformada de Laplace de (3.1) temos:

$$X^*(s) = x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \dots$$

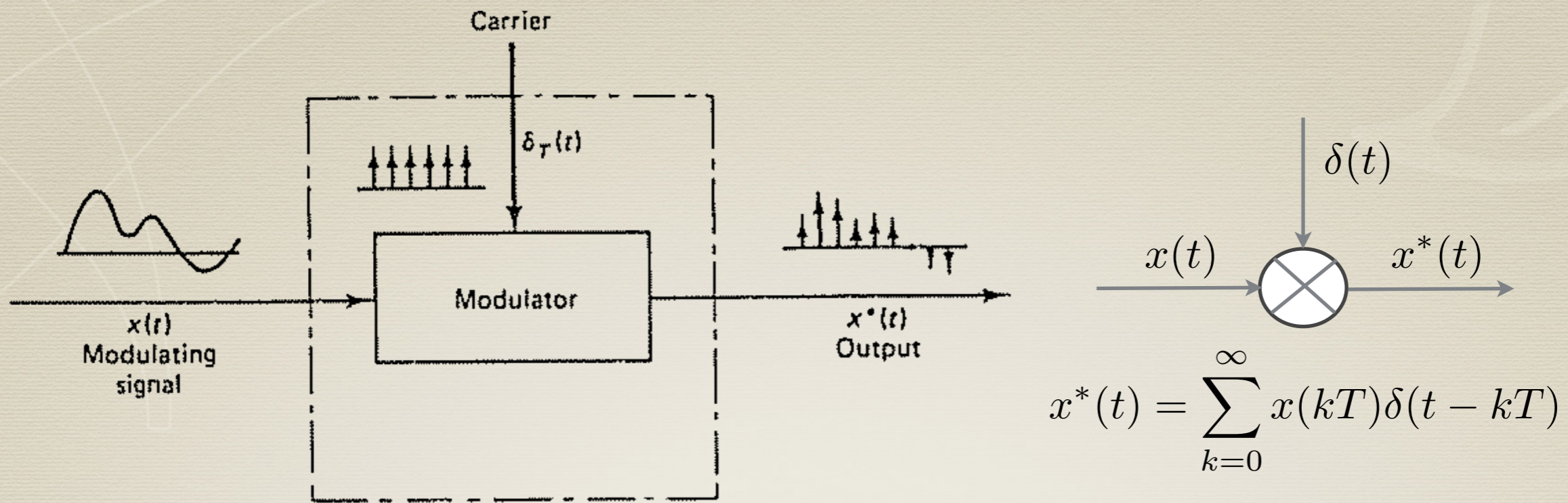


Figure 3-2 Impulse sampler as a modulator.

- Realizando a transformada de Laplace de (3.1) temos:

$$X^*(s) = \mathcal{L}[x^*(t)] = x(0)\mathcal{L}[\delta(t)] + x(T)\mathcal{L}[\delta(t - T)] + x(2T)\mathcal{L}[\delta(t - 2T)] + \dots$$

$$X^*(s) = x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \dots$$

$$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

- Lembrando que: $\mathcal{L}\{\delta(t - kT)\} = e^{-Ts}$

- Pode ser demonstrado que: $X^{*(s)}|_{s=(1/T)\ln z} = X(z)$

- Realizando a transformada de Laplace de (3.1) temos:

$$X^*(s) = \mathcal{L}[x^*(t)] = x(0)\mathcal{L}[\delta(t)] + x(T)\mathcal{L}[\delta(t - T)] + x(2T)\mathcal{L}[\delta(t - 2T)] + \dots$$

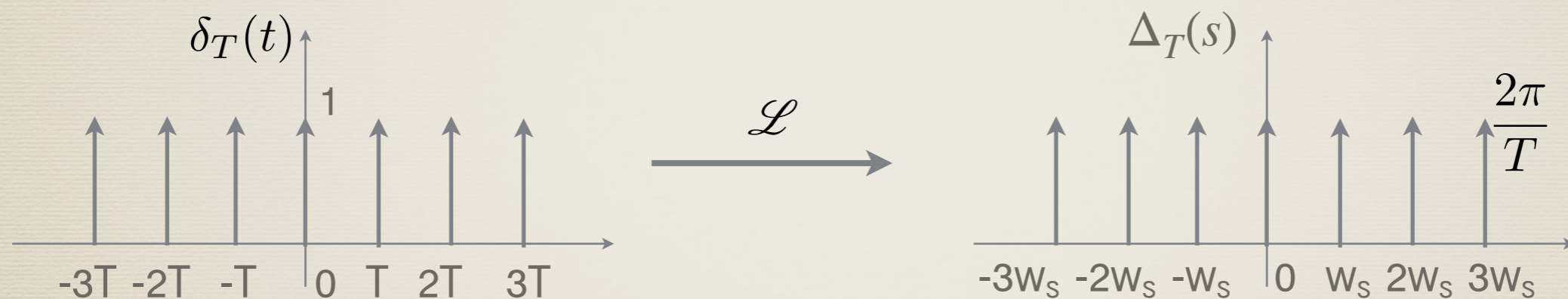
$$X^*(s) = x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \dots$$

$$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$X^{*(s)}|_{s=(1/T)\ln z} = X(z)$$

- Lembrando que: $\mathcal{L}\{\delta(t - kT)\} = e^{-Ts}$

- Então no domínio frequência, o trem de pulsos é visualizado como:



$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

$$T = \frac{1}{f_s}$$

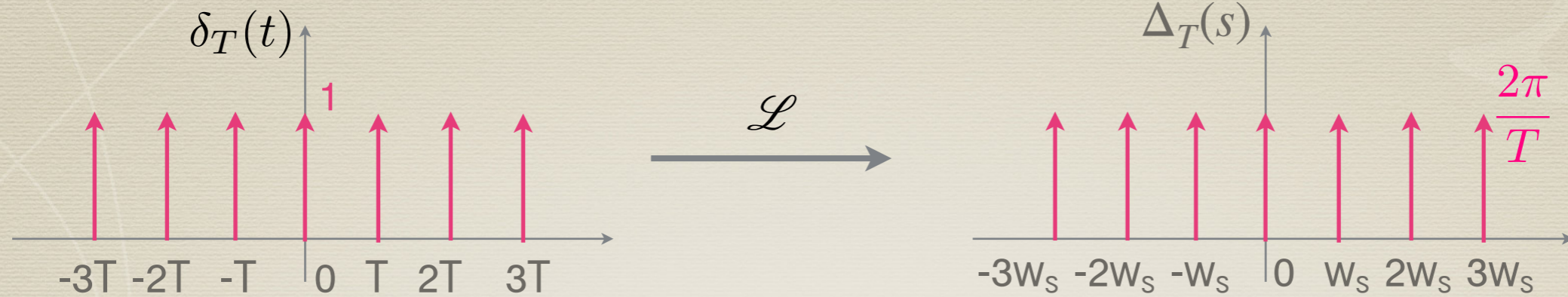
$$\Delta_T(s) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} \cdot e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T}$$

- Como: $x^*(t) = x(t) * \delta_T(t)$ - No Domínio frequência teremos uma convolução:

- Então no domínio frequência, o trem de pulsos é visualizado como:



$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

$$T = \frac{1}{f_s}$$

$$\Delta_T(s) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} \cdot e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T}$$

- Como: $x^*(t) = x(t) * \delta_T(t)$ - No Domínio frequência teremos uma convolução:

$$X^*(s) = \frac{1}{2\pi} [X(s) \cdot \Delta(s)]$$

$$X^*(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\theta) \cdot \Delta(\omega - \theta) d\theta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\theta) \cdot \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - \theta - k\omega_s) d\theta$$

$$X^*(s) = \frac{1}{T} \cdot \sum_{k=-\infty}^{+\infty} X(\omega - k\omega_s)$$

Resultado da amostragem de um sinal por um trem de impulsos

- Como: $x^*(t) = x(t) * \delta_T(t)$ - No Domínio frequência teremos uma convolução:

$$X^*(s) = \frac{1}{2\pi} [X(s) \cdot \Delta(s)]$$

$$X^*(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\theta) \cdot \Delta(\omega - \theta) d\theta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\theta) \cdot \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - \theta - k\omega_s) d\theta$$

$$X^*(s) = \frac{1}{T} \cdot \sum_{k=-\infty}^{+\infty} X(\omega - k\omega_s)$$

Resultado da amostragem
de um sinal por um trem
de impulsos

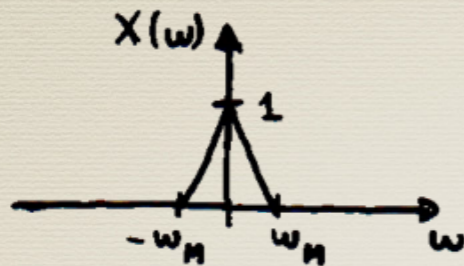
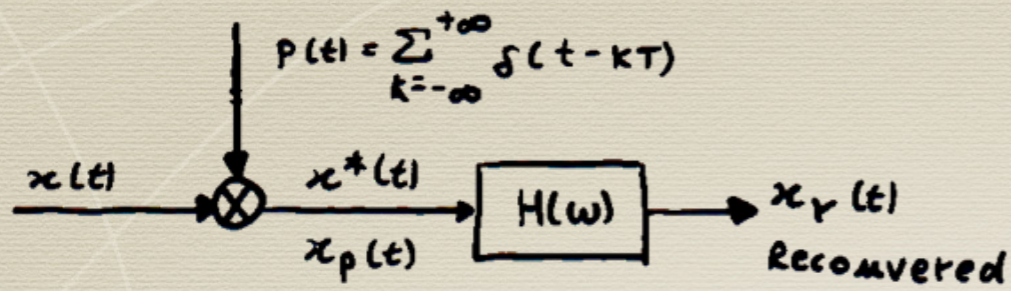
ou:

$$X^*(s) = X^*(s \pm j\omega_s k), \quad k = 0, 1, 2, \dots$$

Note que $X(s)$ possui um pólo em $s=s_1$, já $X^*(s)$ possui pólos múltiplos em:

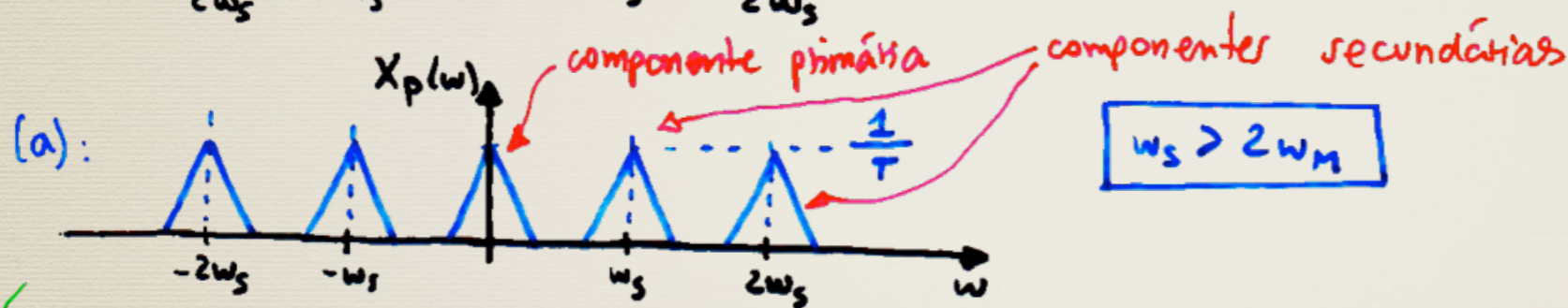
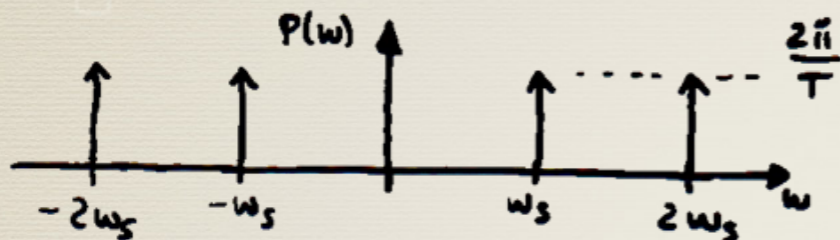
$$s = s_1 \pm j\omega_s k \quad (k = 0, 1, 2, \dots)$$

Observando o espectro resultante, supondo que $X(\omega)$ possui banda limitada.

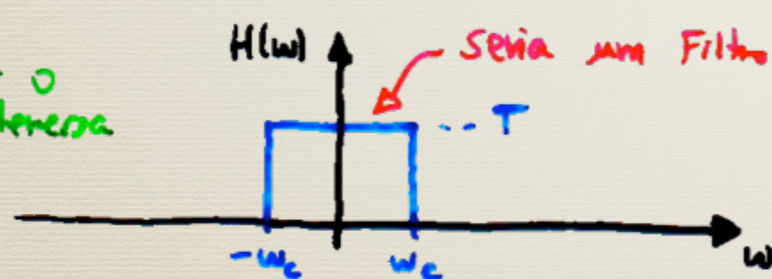


$$x_p(t) = x(t) \cdot p(t)$$

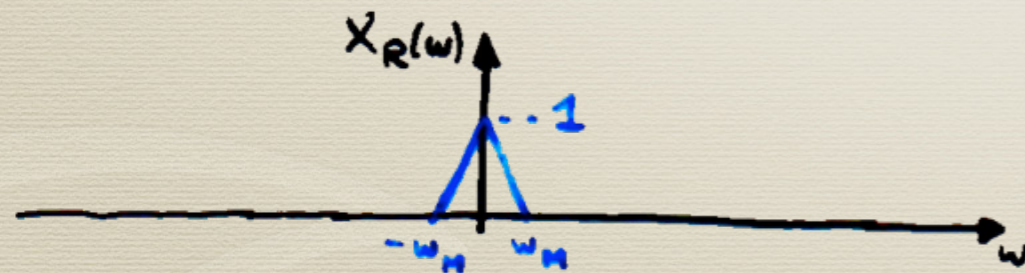
$$X_p(\omega) = \frac{1}{2\pi} [X(\omega) * P(\omega)]$$



↳ p/recomp o sinal, no interessa apenas a componente principal



$$w_M < w_c < (w_s - w_M)$$



Note:

Para não haver recobrimento de espectros em (a), temos que ter:

$$w_s > 2w_M$$

- Teorema da Amostragem (Shanon):

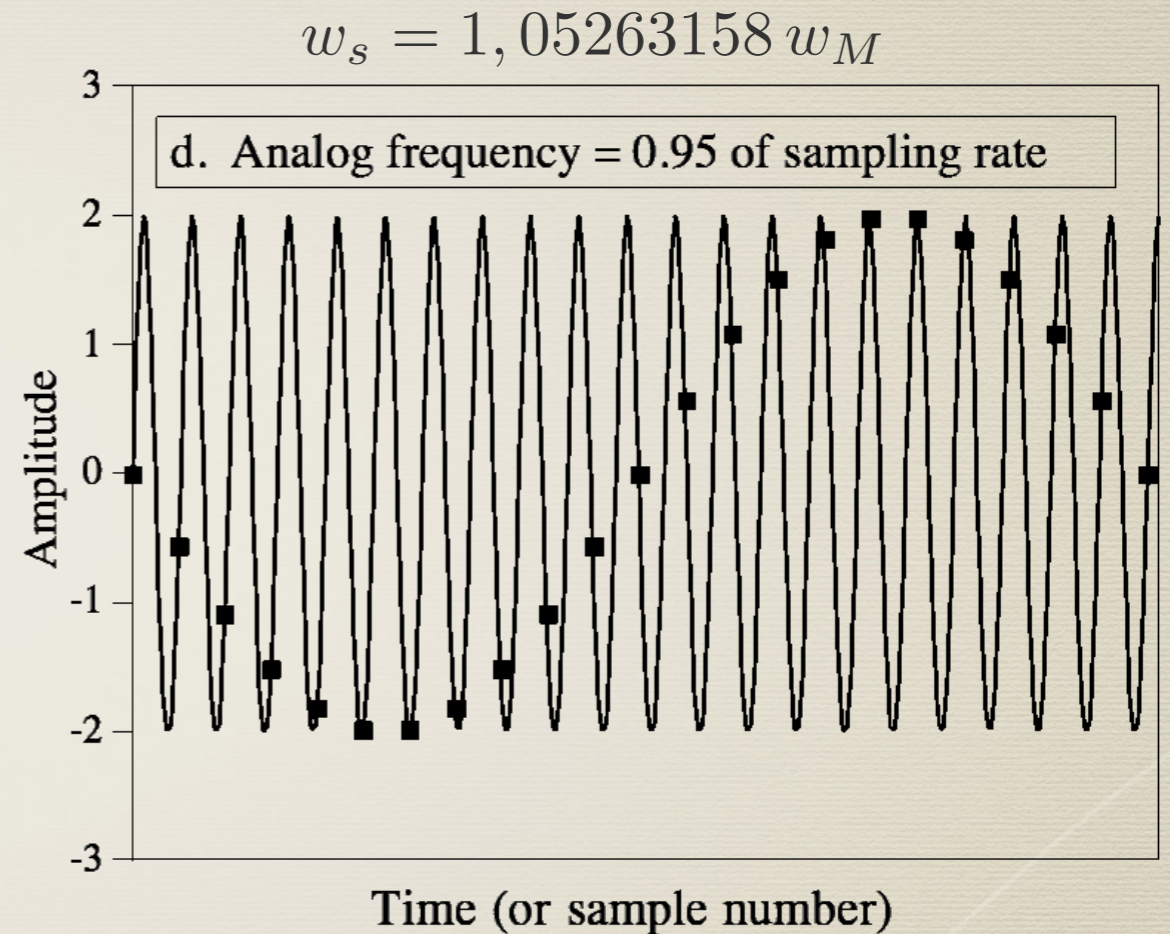
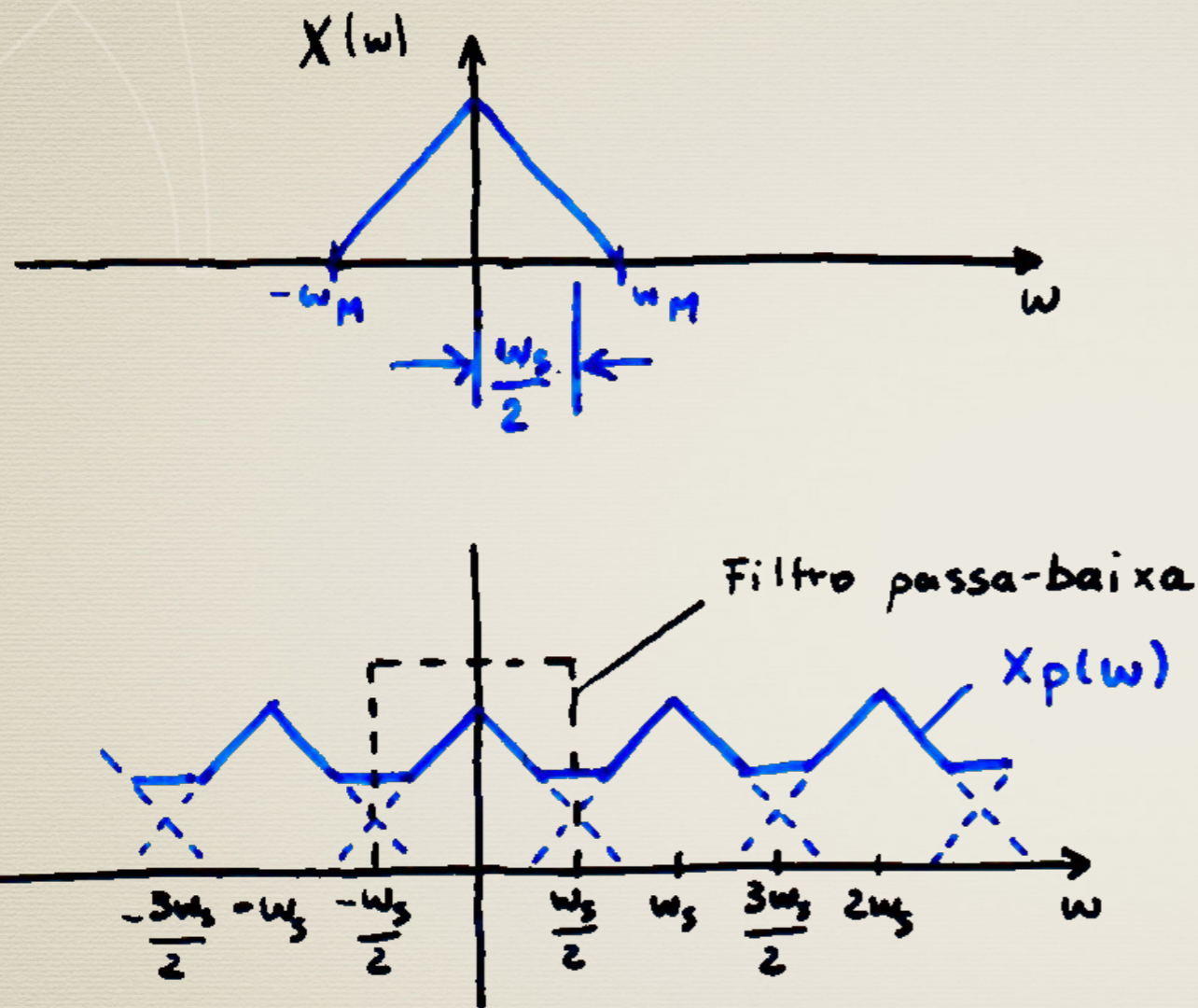
$$w_s = \frac{2\pi}{T} > 2w_M$$

- Freq. (taxa) de Nyquist:

$$w_s = 2w_M$$

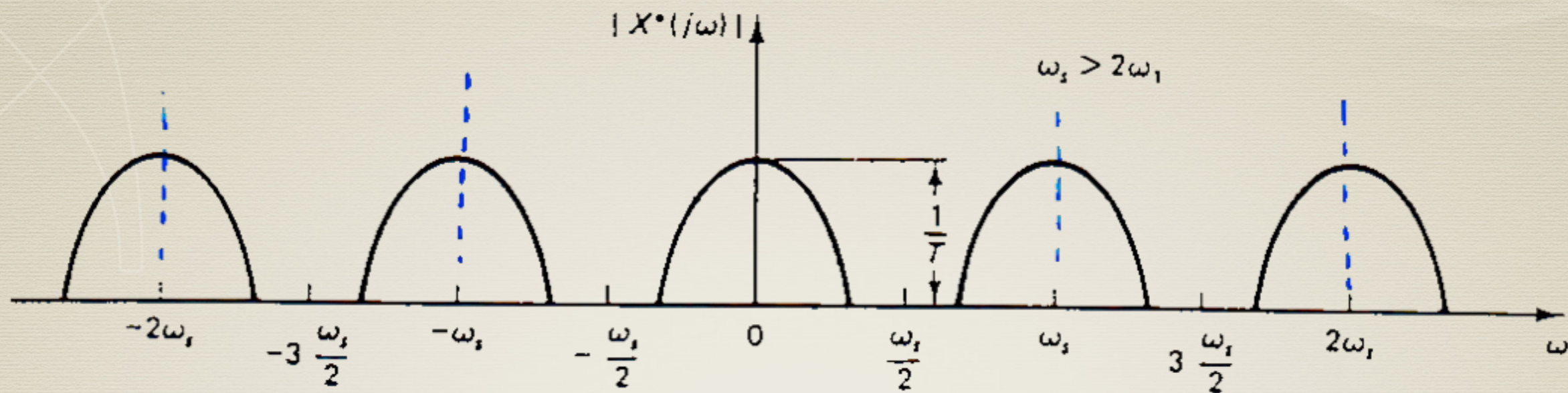
Fig. 2.5 - Amostragem por Impulsor (ideal)

Caso de Sub-amostragem (aliasing)



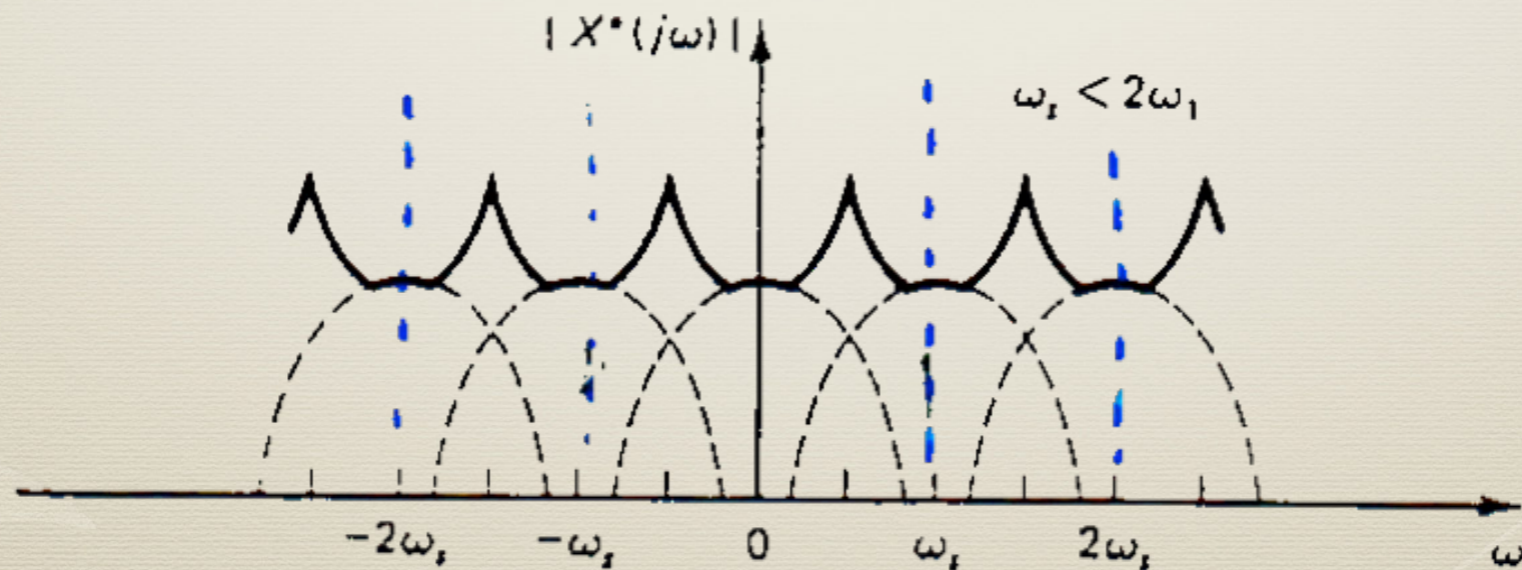
Outros Casos

$\omega_s > 2\omega_1$:



(a)

$\omega_s < 2\omega_1$:



Efeitos: "Hidden Oscillation"

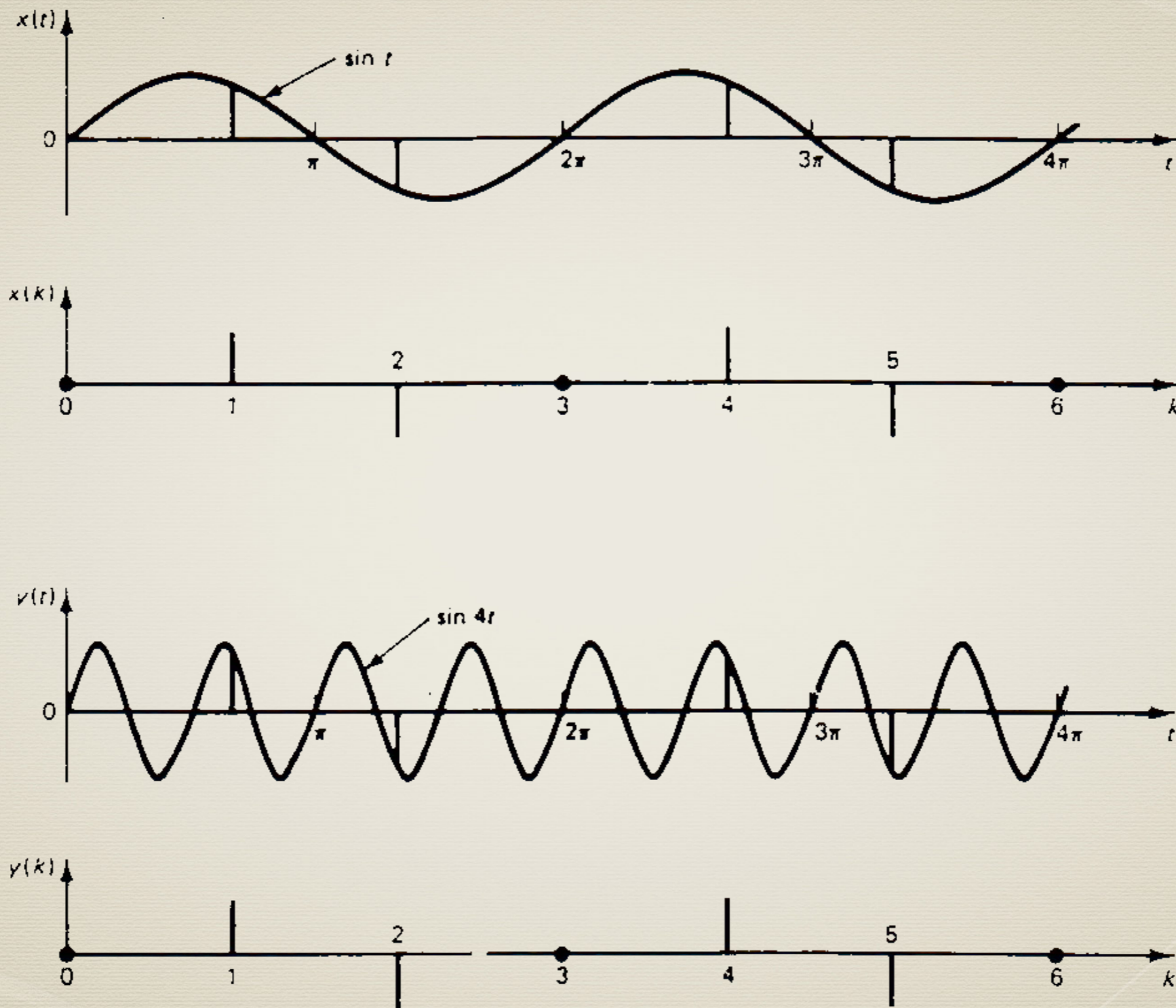
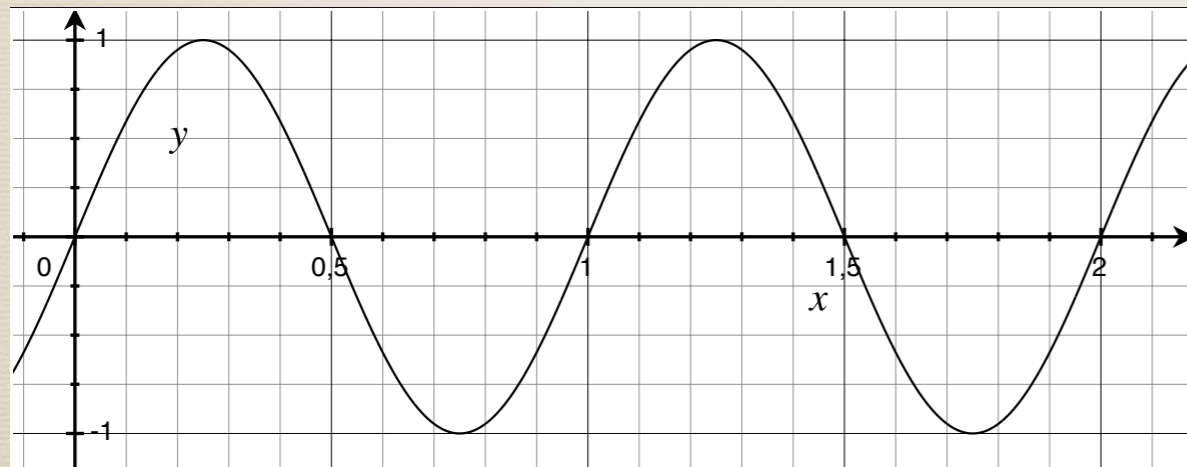


Figure 3-18 Plots of signals $x(t) = \sin t$ and $y(t) = \sin 4t$ and their sampled signals. $\omega_s = 3$ rad/s.

Exercício de Sub-amostragem

- * Seja uma onda senoidal de 1,0 Vp oscilando à 1 Hz.
Sua equação seria: $y(t) = 1 \cdot \sin(2\pi \cdot 1 \cdot t)$
Lembrando que: $\omega = 2\pi f$.
- * O que acontece se este sinal for amostrado à 2 Hz? Recorde-se que estaríamos respeitando o teorema de Nyquist.



Obs.: Este exercício pode ser retomado ou re-estudado depois de estudada a transformada-Z de uma uma onda senoidal:

$$\mathcal{Z} \{ \sin(\omega t) \} = \frac{z \sin(\omega T)}{z^2 - 2z \cos(\omega T) + 1}$$

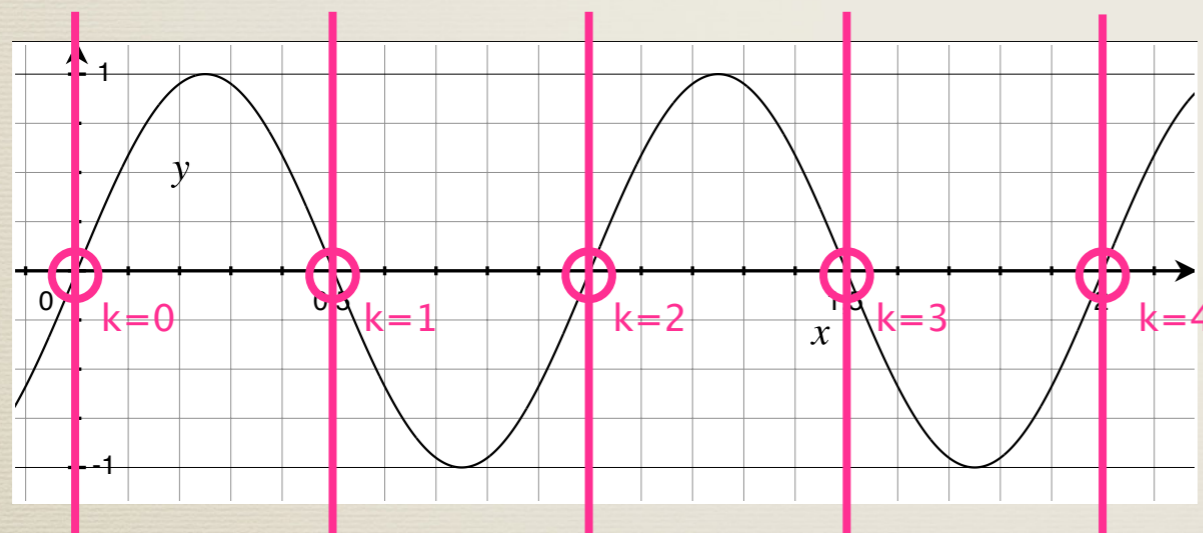
Exercício de Sub-amostragem

* Seja uma onda senoidal de 1,0 Vp oscilando à 1 Hz.

Sua equação seria: $y(t) = 1 \cdot \sin(2\pi \cdot 1 \cdot t)$

Lembrando que: $\omega = 2\pi f$.

* O que acontece se este sinal for amostrado à 2 Hz? Recorde-se que estaríamos respeitando o teorema de Nyquist.



Obs.: Este exercício pode ser retomado ou re-estudado usando a transformada-Z de uma onda senoidal:

$$\mathcal{Z} \{ \sin(\omega t) \} = \frac{z \sin(\omega T)}{z^2 - 2z \cos(\omega T) + 1}$$

Note:

$$y^*(t) = y[kT] = \sin(2\pi \cdot kT)$$

$$T = 1/f_s \therefore T = 1/2 \text{ (} f_s = 2 \text{ Hz)}$$

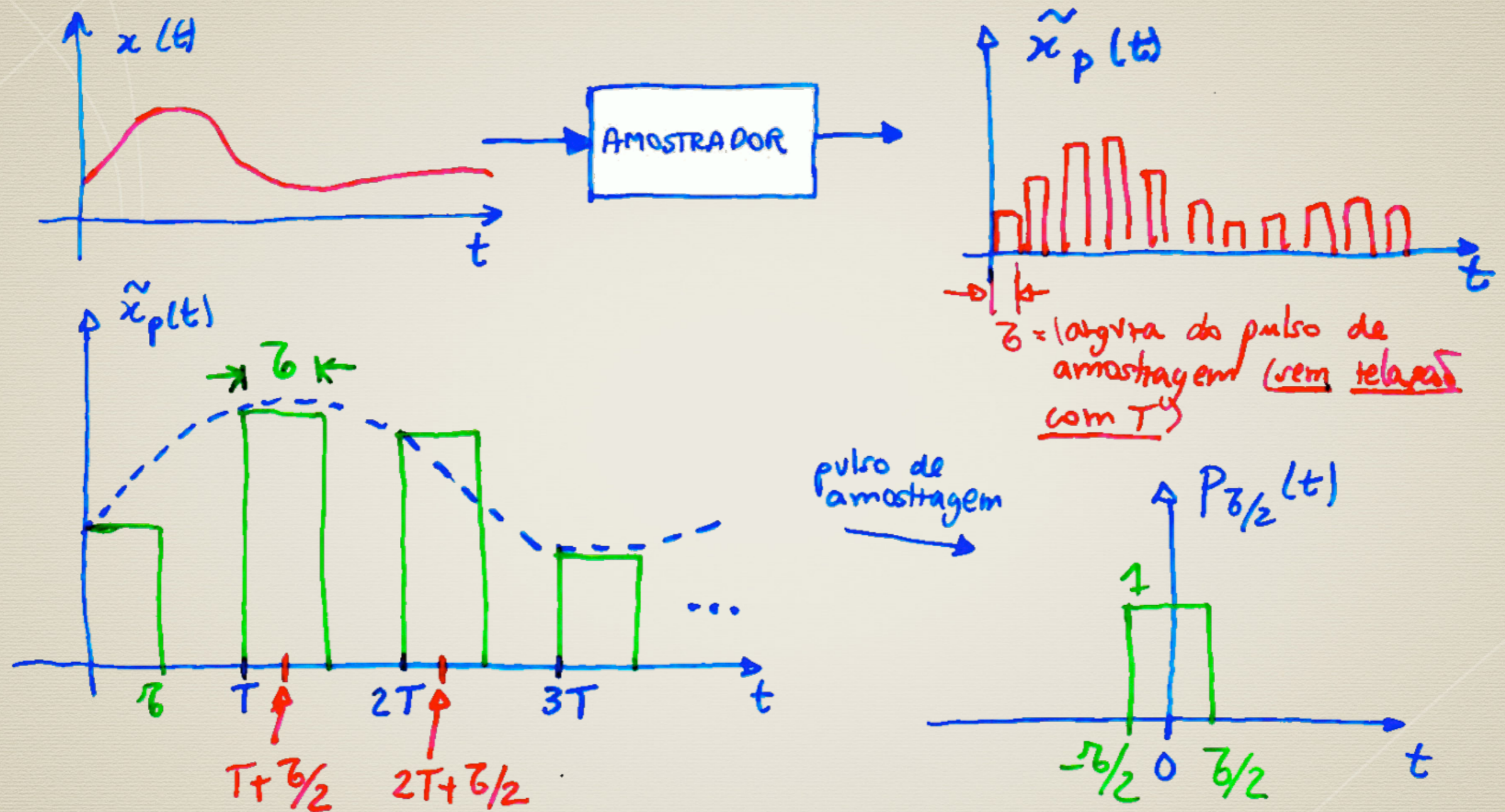
$$y[kT] = \sin\left(\frac{2\pi k}{2}\right) = \sin(k\pi)$$

k	y[kT]
0	0
1	0
2	0
3	0
4	0
⋮	0

Exemplo de Subamostragem

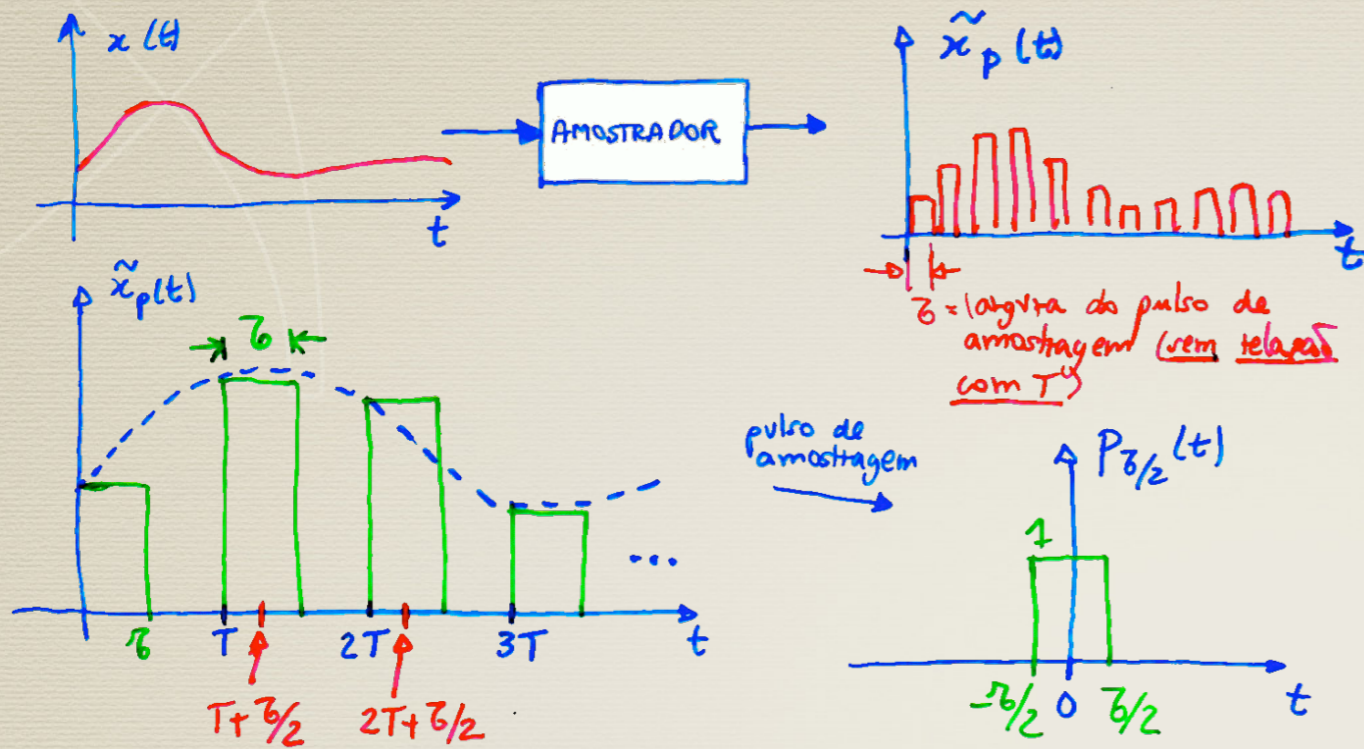
- * Suponha agora que a onda sinusoidal de 1 Hz seja amostrada à 2 Hz.
- * O que acontecerá ?

Amostragem por trem de pulsos



- na prática:
- amostragem realizada por pulsos;
 - filtro passa-baixa ideal não existe (não realizável fisicamente)

Amostragem por trem de pulsos



Nova modelagem:

$$\tilde{x}_p(t) = \sum_{k=-\infty}^{+\infty} x(kT) \cdot p_{\tau/2}(t - kT - \tau/2)$$

somatório de pulsos deslocados no tempo:

$$\tilde{X}_p(\omega) = \sum_{k=-\infty}^{+\infty} x(kT) \cdot \mathcal{F}\left\{p_{\tau/2}(t - kT - \tau/2)\right\}$$

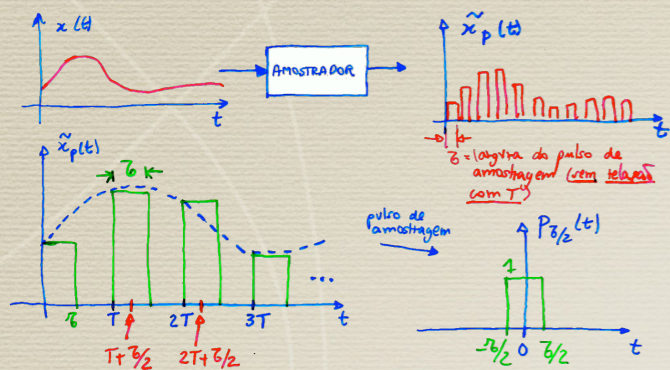
$$\mathcal{F}\left\{p_{\tau/2}(t)\right\} = \tau \cdot \frac{\sin(\omega \tau/2)}{\frac{\omega \tau}{2}}$$

Função sampling

$$\sum_{k=-\infty}^{+\infty} p_{\tau/2}(t - kT - \tau/2)$$

Diagram showing the multiplication of the input signal $x(t)$ by the sampling pulse train to produce the sampled signal $\tilde{x}_p(t)$.

Amostragem por trem de pulsos



$$\tilde{x}_p(t) = \sum_{n=-\infty}^{+\infty} x(kT) \cdot p_{b/2}(t - kT - \frac{b}{2})$$

$$\therefore x(t) \otimes \sum_{k=-\infty}^{+\infty} p_{b/2}(t - kT - \frac{b}{2}) \rightarrow \tilde{x}_p(t)$$

somatório de pulsos deslocados no tempo:

$$\tilde{X}_p(\omega) = \sum_{k=-\infty}^{+\infty} x(kT) \cdot \mathcal{F}\left\{p_{b/2}(t - kT - \frac{b}{2})\right\}$$

$$\mathcal{F}\left\{p_{b/2}(t)\right\} = b \cdot \frac{\sin(\omega b/2)}{\frac{\omega b}{2}}$$

Função sampling

Nova modelagem:

$$\mathcal{F}\left\{p_{b/2}(t - kT - \frac{b}{2})\right\} = b \cdot \frac{\sin(\omega b/2)}{\omega b/2} \cdot e^{-j\omega(kT - \frac{b}{2})}$$

deslocamento de fase

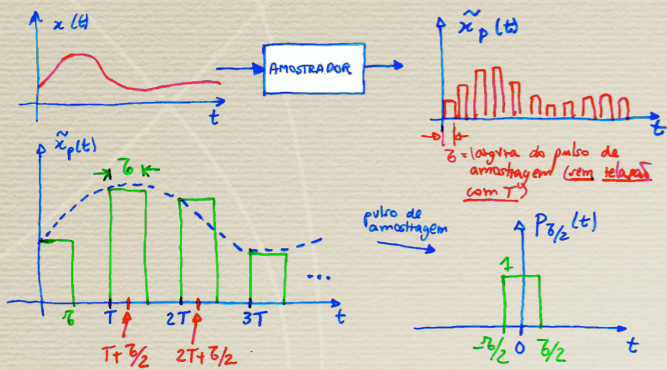
$$= b \cdot \frac{\sin(\omega b/2)}{\omega b/2} \cdot e^{-j\omega b/2} \cdot e^{-j\omega kT}$$

$$\tilde{X}_p(\omega) = b \cdot \frac{\sin(\omega b/2)}{\omega b/2} \cdot e^{-j\omega b/2} \cdot \sum_{k=-\infty}^{+\infty} x(kT) \cdot e^{-j\omega kT}$$

não depende de $x(t)$;
não depende de f_s .

$X_p(\omega)$

Amostragem por trem de pulsos



$$\mathcal{F}\{p_{b/2}(t - kT - b/2)\} = b \cdot \frac{\sin(\omega b/2)}{\omega b/2} \cdot e^{-j\omega(kT - b/2)}$$

deslocamento de fase

$$= b \cdot \frac{\sin(\omega b/2)}{\omega b/2} \cdot e^{-j\omega b/2} \cdot e^{-j\omega kT}$$

$$\tilde{X}_p(\omega) = \underbrace{b \cdot \frac{\sin(\omega b/2)}{\omega b/2} \cdot e^{-j\omega b/2}}_{\substack{\text{n\~ao depende de } x(t); \\ \text{n\~ao depende de } f_s.}} \cdot \underbrace{\sum_{k=-\infty}^{\infty} x(kT) \cdot e^{-j\omega kT}}_{X_p(\omega)}$$

$$\tilde{X}_p(\omega) = \underbrace{b \cdot \frac{\sin(\omega b/2)}{\omega b/2} \cdot e^{-j\omega b/2}}_{\substack{\text{distor\~ao em fase e amplitude}}} \cdot X_p(\omega)$$

Nova modelagem:

\therefore distor\~ao em fase e amplitude

$$e^{-j\omega b/2}$$

↑

devido ao atraso no tempo, de $b/2$

$$\frac{b \cdot \sin(\omega b/2)}{\omega b/2}$$

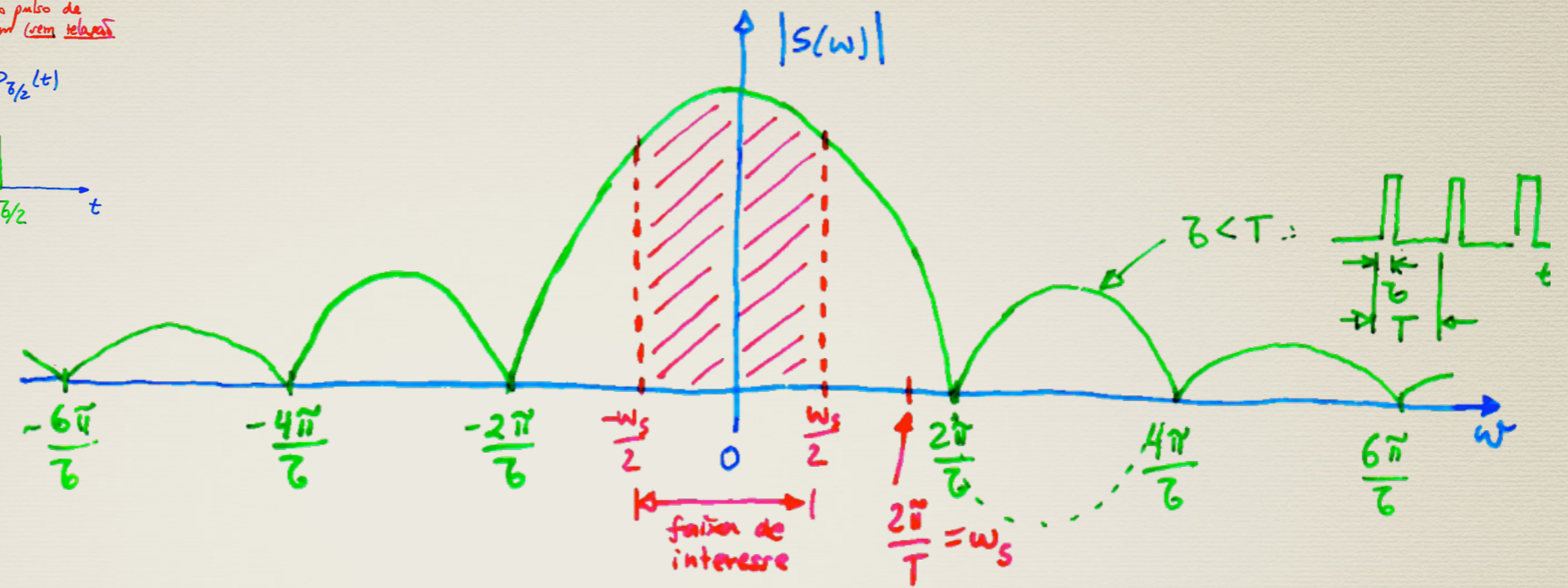
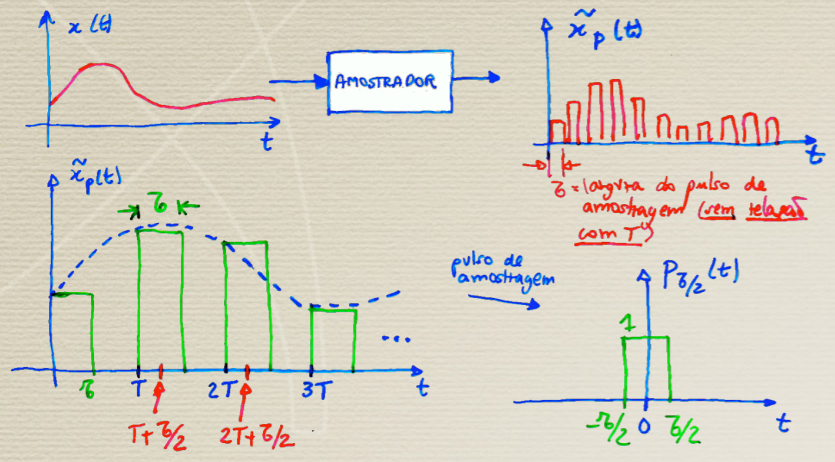
$$\tilde{X}_p(\omega) = b \cdot s(\omega) \cdot X_p(\omega)$$

$$\text{onde } s(\omega) = \frac{\sin(\omega b/2)}{\omega b/2} \cdot e^{-j\omega b/2}$$

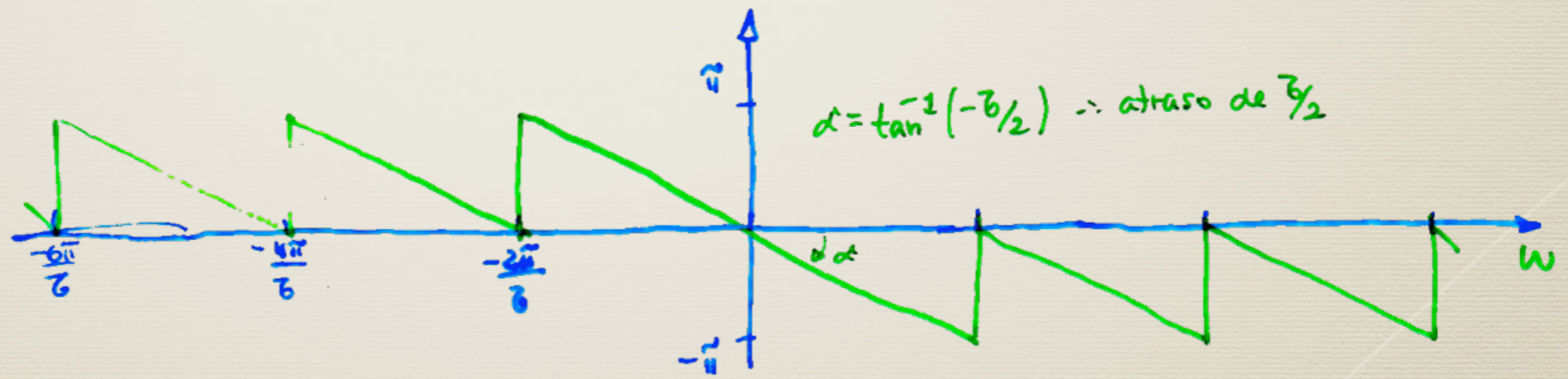
$$\Leftarrow \text{zeros em } \frac{\omega b}{2} = k \cdot \pi \quad (k \neq 0)$$

$$\text{ou em } \omega = k \cdot \frac{2\pi}{b} \quad (k \neq 0)$$

Amostragem por trem de pulsos



Nova modelagem:



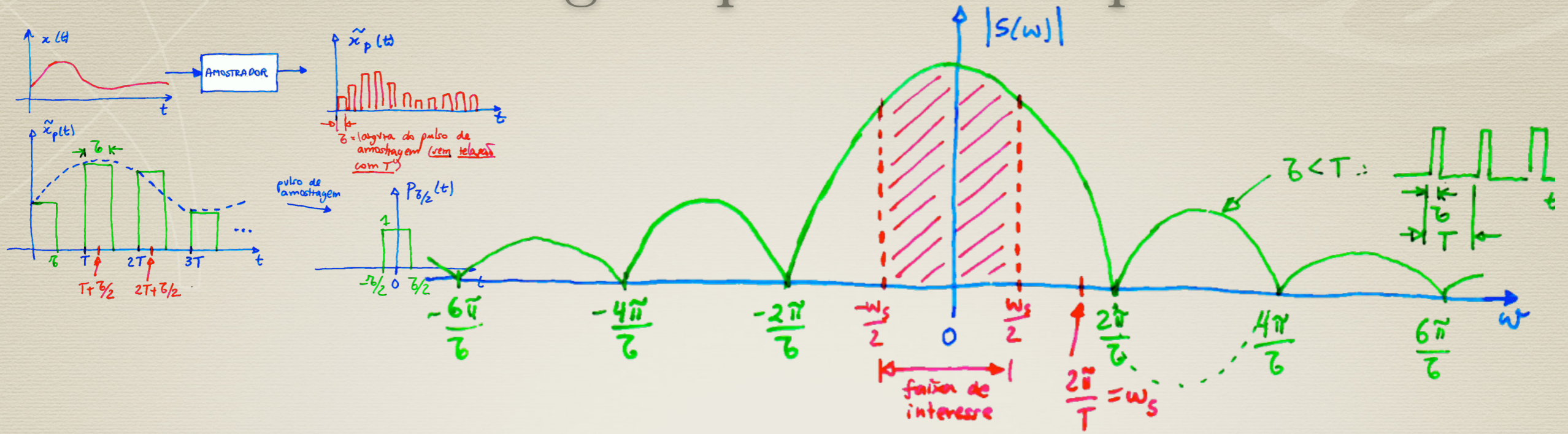
$$\tilde{X}_p(w) = b \cdot \frac{\sin(wb/2)}{wb/2} \cdot e^{-jwb/2} \cdot X_p(w)$$

\therefore distorção em fase e amplitude
 $e^{-jwb/2}$ devido ao atraso no tempo, de $b/2$
 $\frac{\sin(wb/2)}{wb/2}$

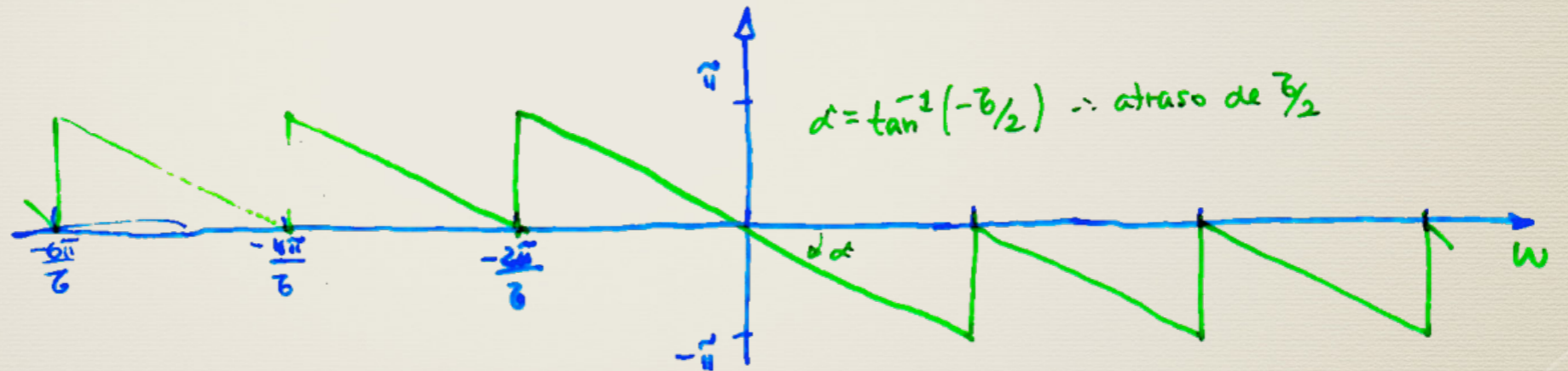
$$\tilde{X}_p(w) = b \cdot S(w) \cdot X_p(w)$$

onde $S(w) = \frac{\sin(wb/2)}{wb/2} \cdot e^{-jwb/2} \iff$ zero em $\frac{wb}{2} = k \cdot \pi \quad (k \neq 0)$
 ou em $w = k \cdot \frac{2\pi}{b} \quad (k \neq 0)$

Amostragem por trem de pulsos



Nova modelagem:



$$\hat{X}_p(\omega) = b \cdot \frac{\sin(\omega b/2)}{\omega b/2} \cdot e^{-j\omega b/2} \cdot X_p(\omega)$$

\therefore distorção em fase e amplitude

$e^{-j\omega b/2}$ devido ao atraso no tempo, de $b/2$

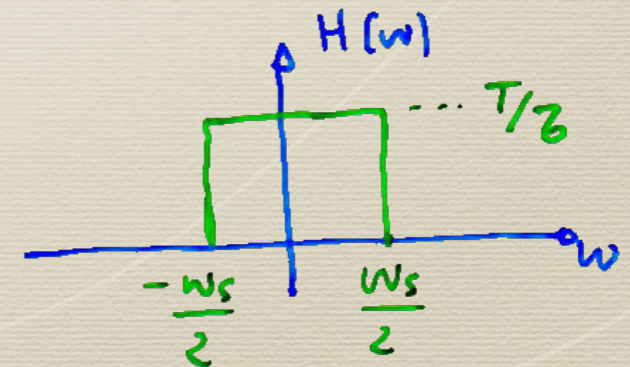
$\frac{b \cdot \sin(\omega b/2)}{\omega b/2}$

$$\hat{X}_p(\omega) = b \cdot S(\omega) \cdot X_p(\omega)$$

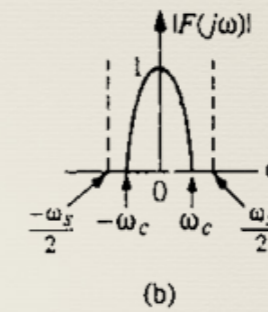
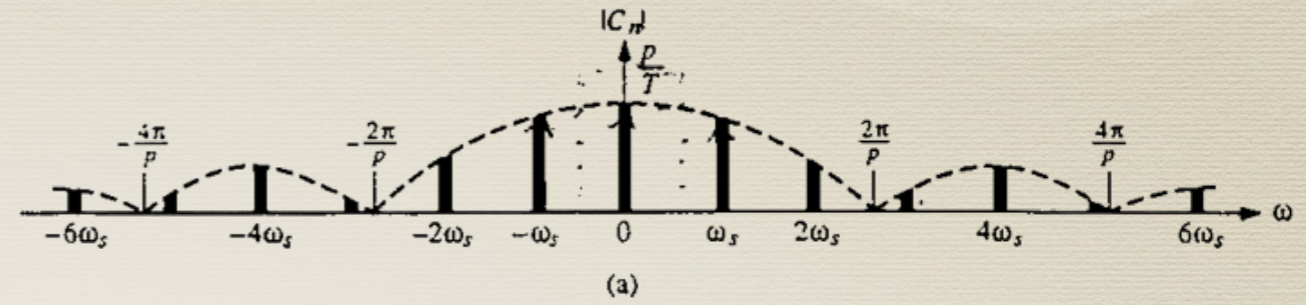
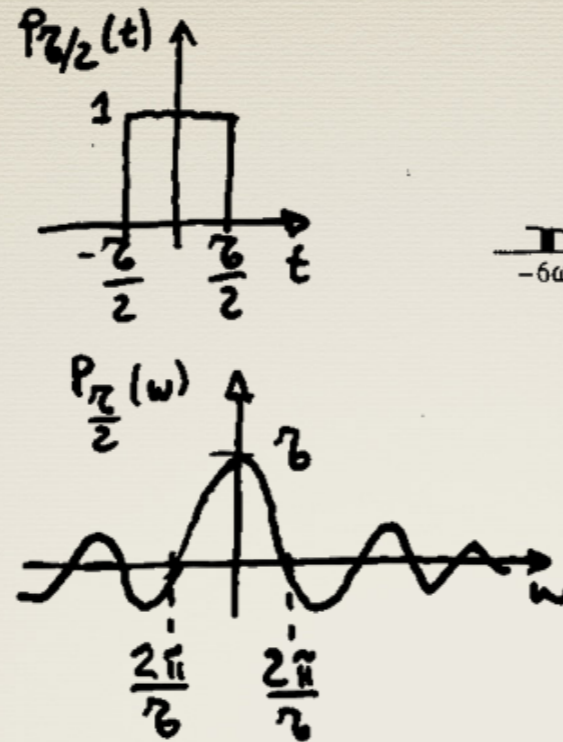
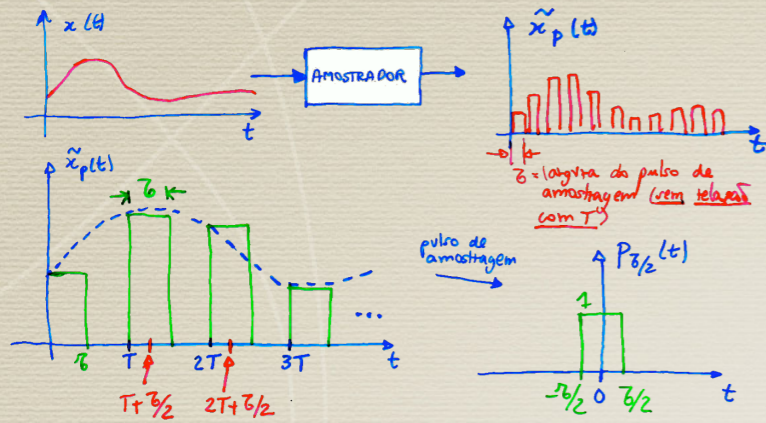
onde $S(\omega) = \frac{\sin(\omega b/2)}{\omega b/2} \leftarrow$ zero em $\frac{\omega b}{2} = k \cdot \pi \quad (k \neq 0)$
 ou em $\omega = k \cdot \frac{2\pi}{b} \quad (k \neq 0)$

Obs.: Posso passar $\hat{X}_p(\omega)$ por um filtro passa-baixas ideal: $\omega_c = \frac{\omega_s}{2}$

Sinal reconstituído

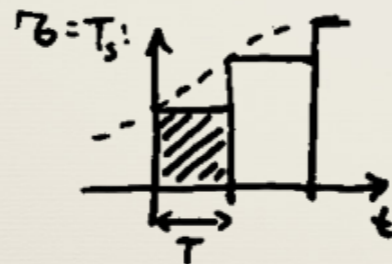


Amostragem por trem de pulsos

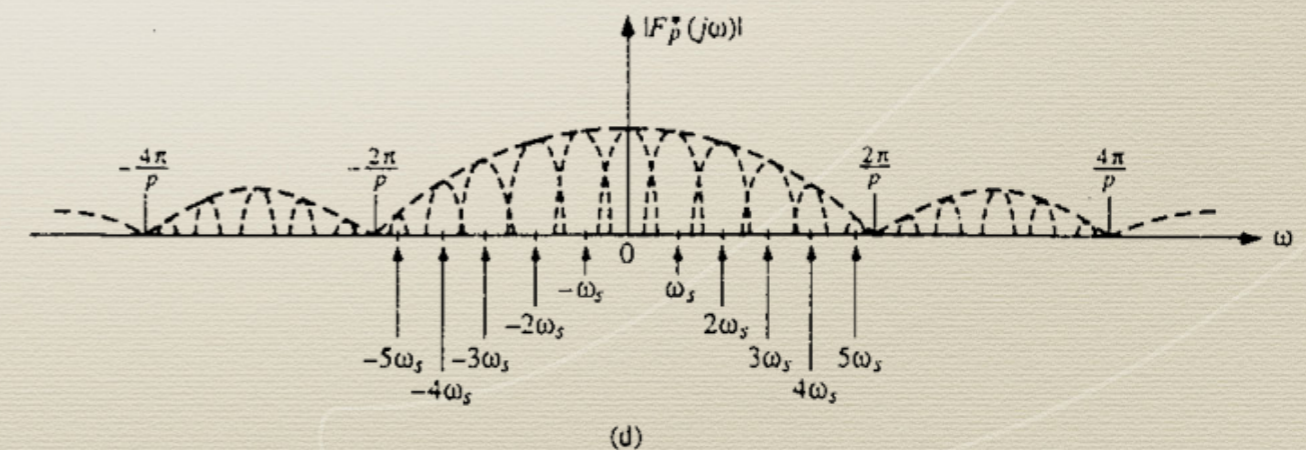
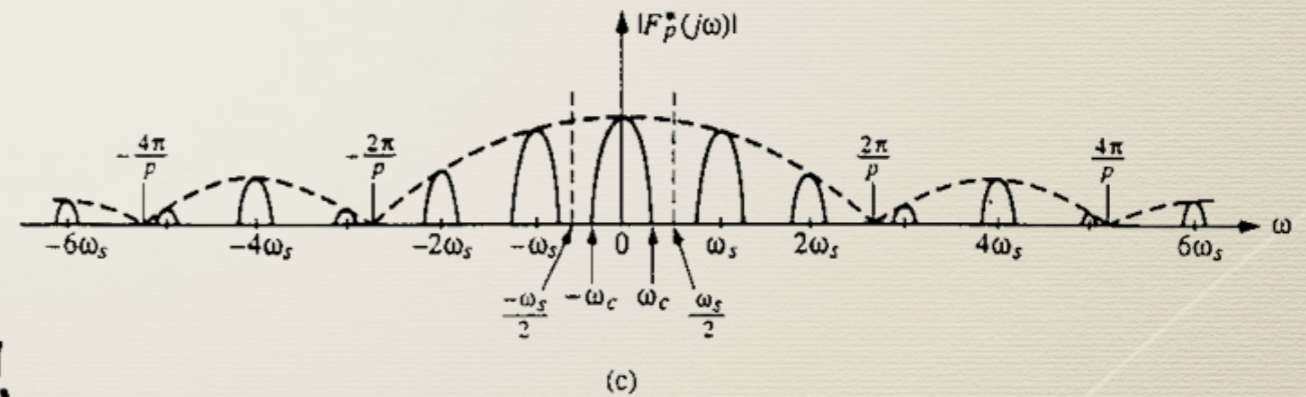
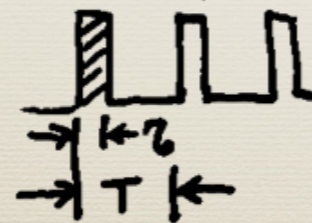


Espectro de amplitudes dos sinais de entrada e sa\u00edda de um amostrador de largura de pulso finita.

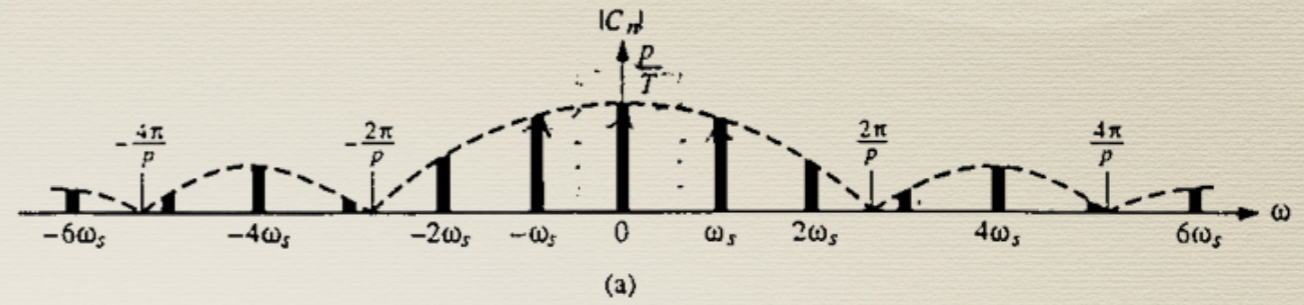
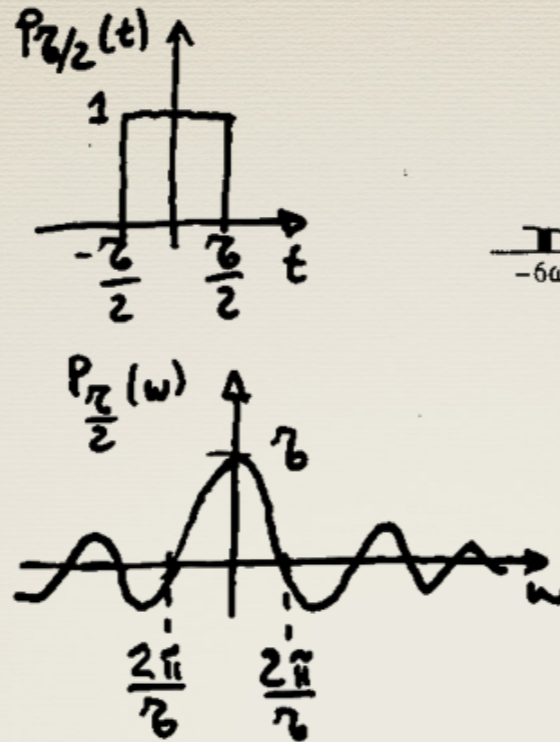
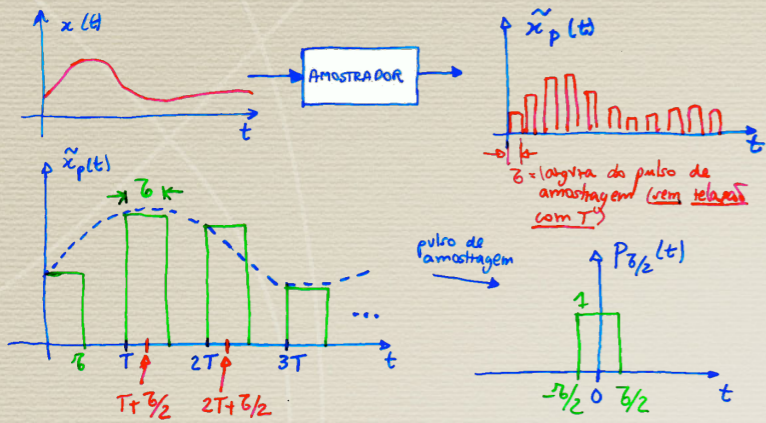
- Espectro de amplitude de um trem unit\u00e1rio de pulsos, $p(t)$;
- Espectro de amplitudes de um sinal cont\u00ednuo, $f(t)$;
- Espectro de amplitude da sa\u00edda amostrada ($\omega_s > 2\omega_c$);
- Espectro de amplitude da sa\u00edda amostrada ($\omega_s < 2\omega_c$).



$T = \tau$: Sample-and-Hold de 100% (ZOH)

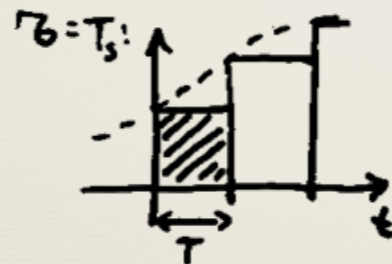
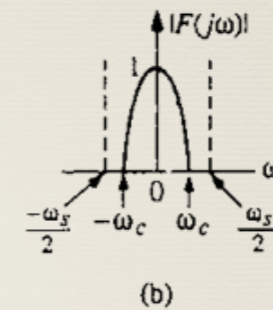


Amostragem por trem de pulsos

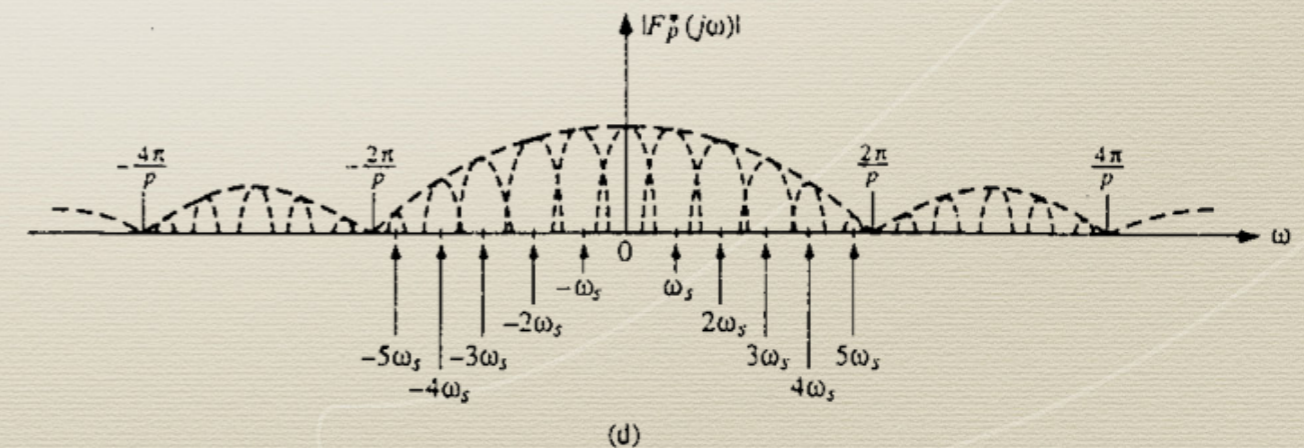
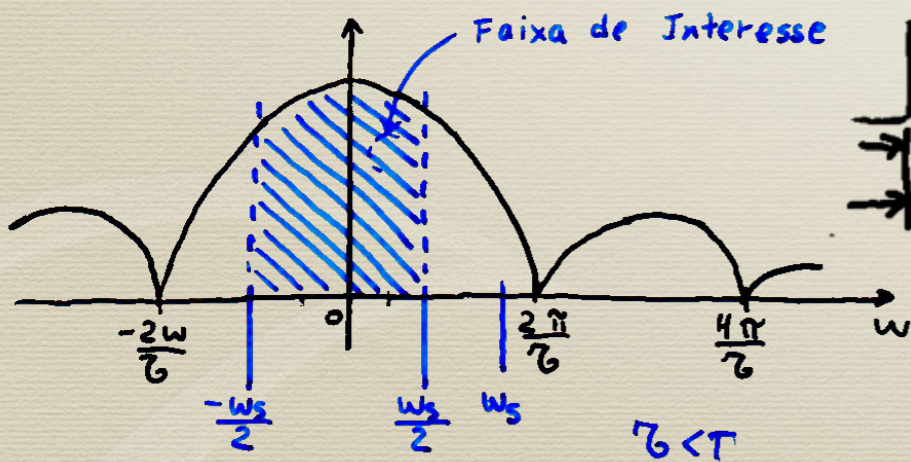
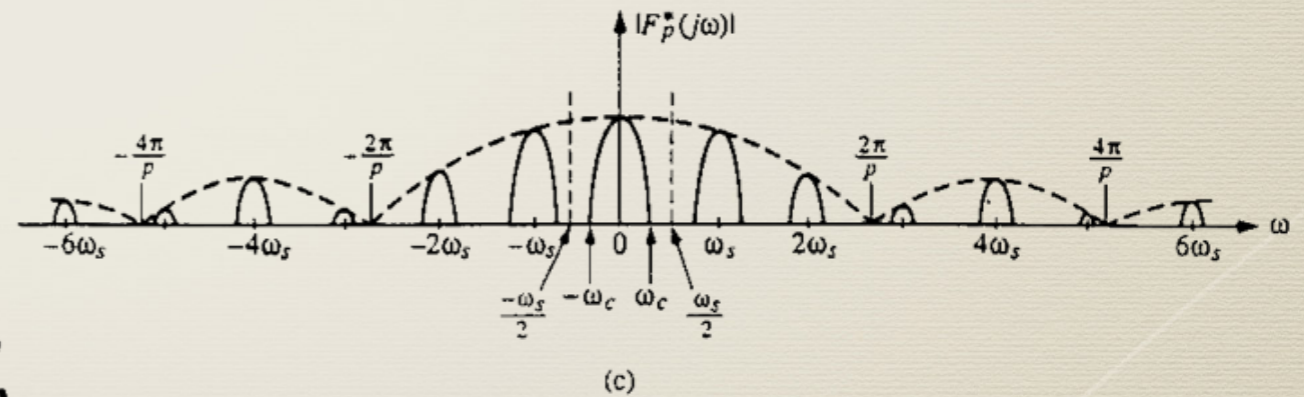


Espectro de amplitudes dos sinais de entrada e saída de um amostrador de largura de pulso finita.

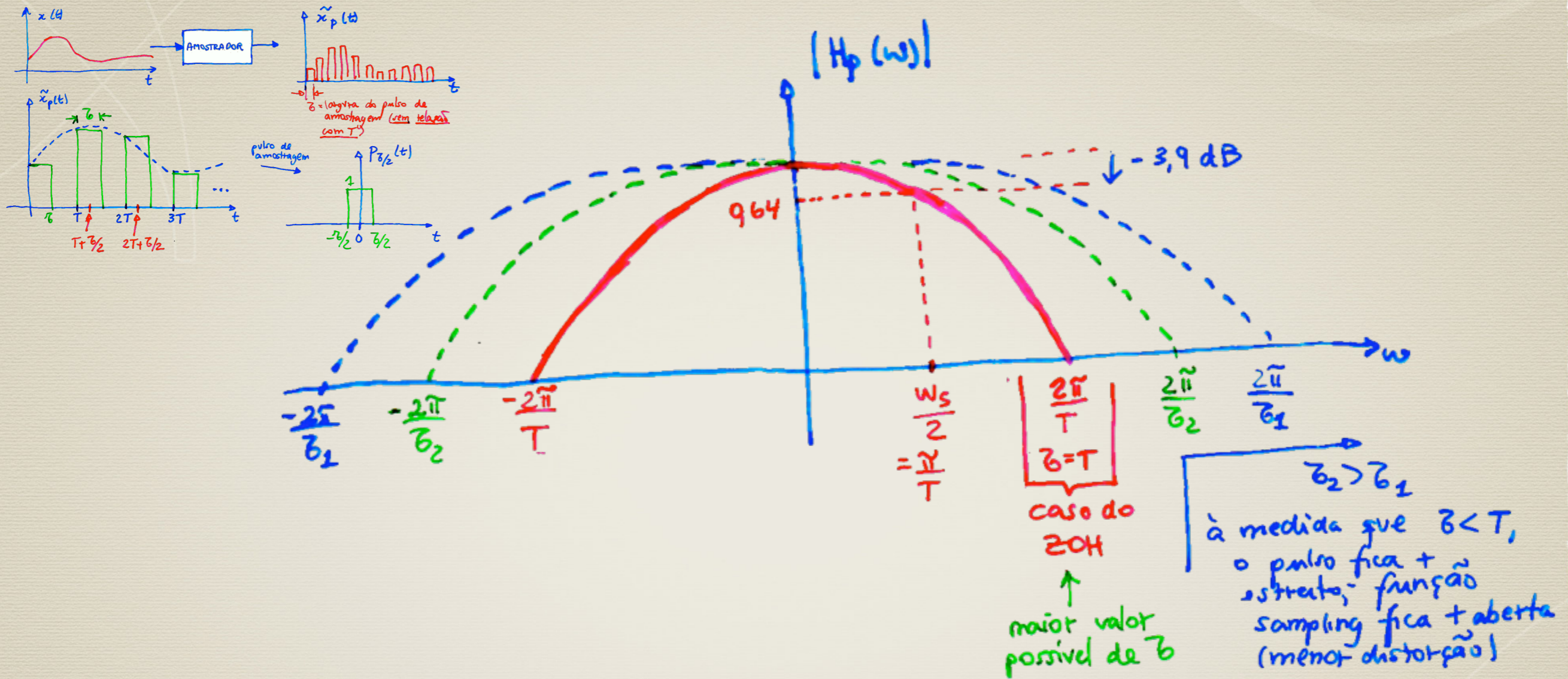
- a) Espectro de amplitude de um trem unitário de pulsos, $p(t)$;
- b) Espectro de amplitudes de um sinal contínuo, $f(t)$;
- c) Espectro de amplitude da saída amostrada ($\omega_s > 2\omega_c$);
- d) Espectro de amplitude da saída amostrada ($\omega_s < 2\omega_c$).



$T = \tau$: Sample-and-Hold de 100% (ZOH)



Distorção na Faixa de Interesse



Quando: $\tau = T$ \therefore "sample-and-Hold" de ordem zero (Z.O.H.)
(100% do ciclo do trabalho)

- embora $\tau = T$ gere o maior erro, é o mais usado por ser o mais simples de ser implementado, então:

$$\tilde{X}_p(\omega) = \frac{\tau}{T} \cdot S(\omega) \cdot \sum X(\omega - k \cdot \omega_s)$$

o fator de atenuação do espectro.

Sample-and-Holder

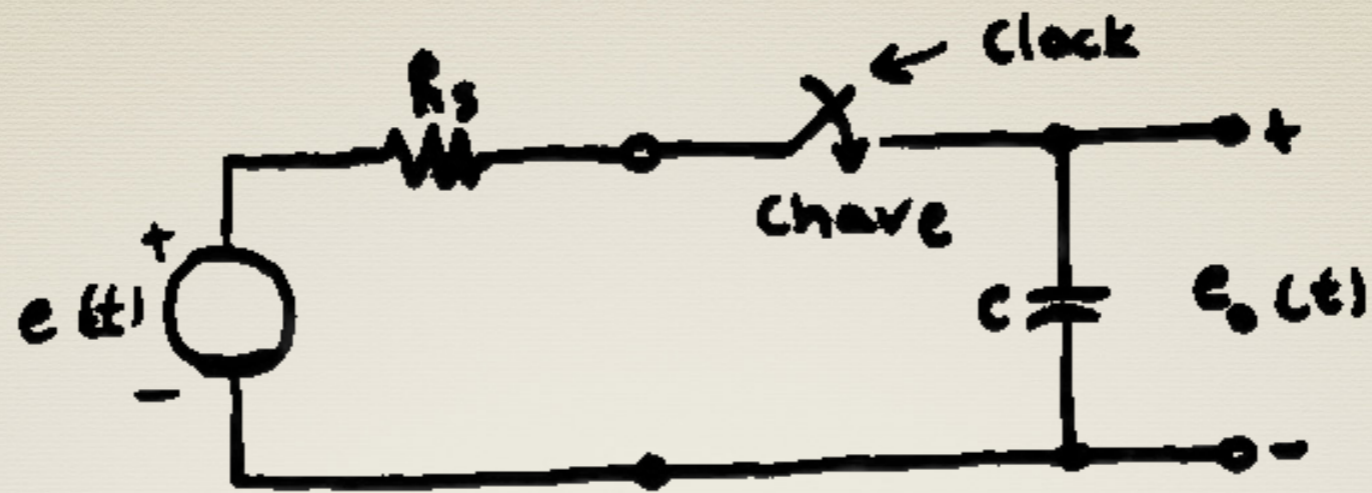


Fig. 2.8 - Circuito simples ilustrando o princípio de "sample-and-hold".

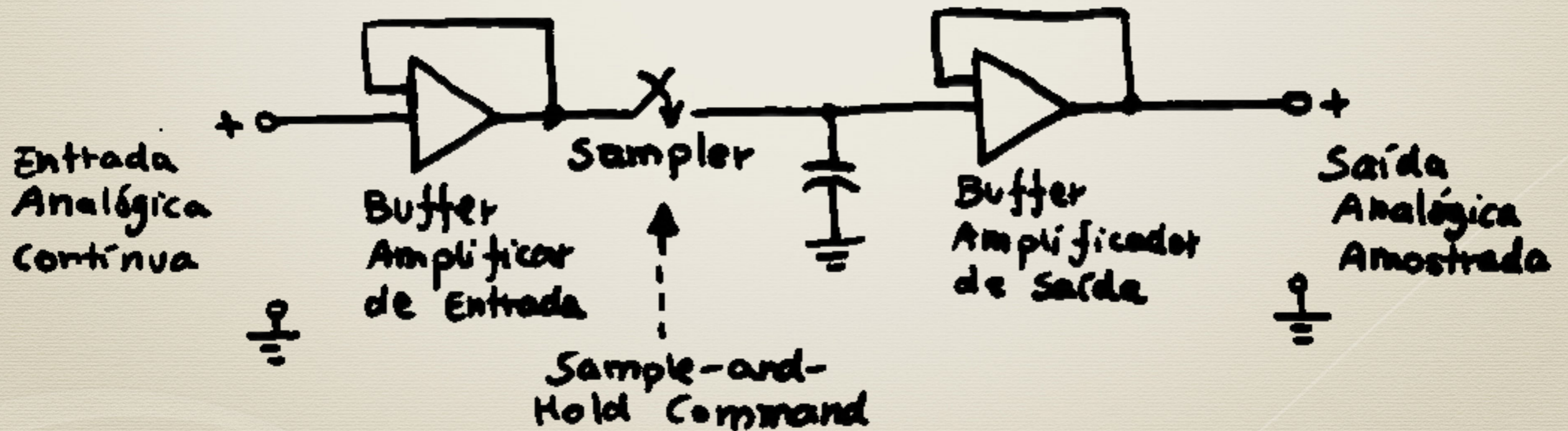


Fig. 2.9 - Dispositivo de "sample-and-hold".

Sample-and-Holder

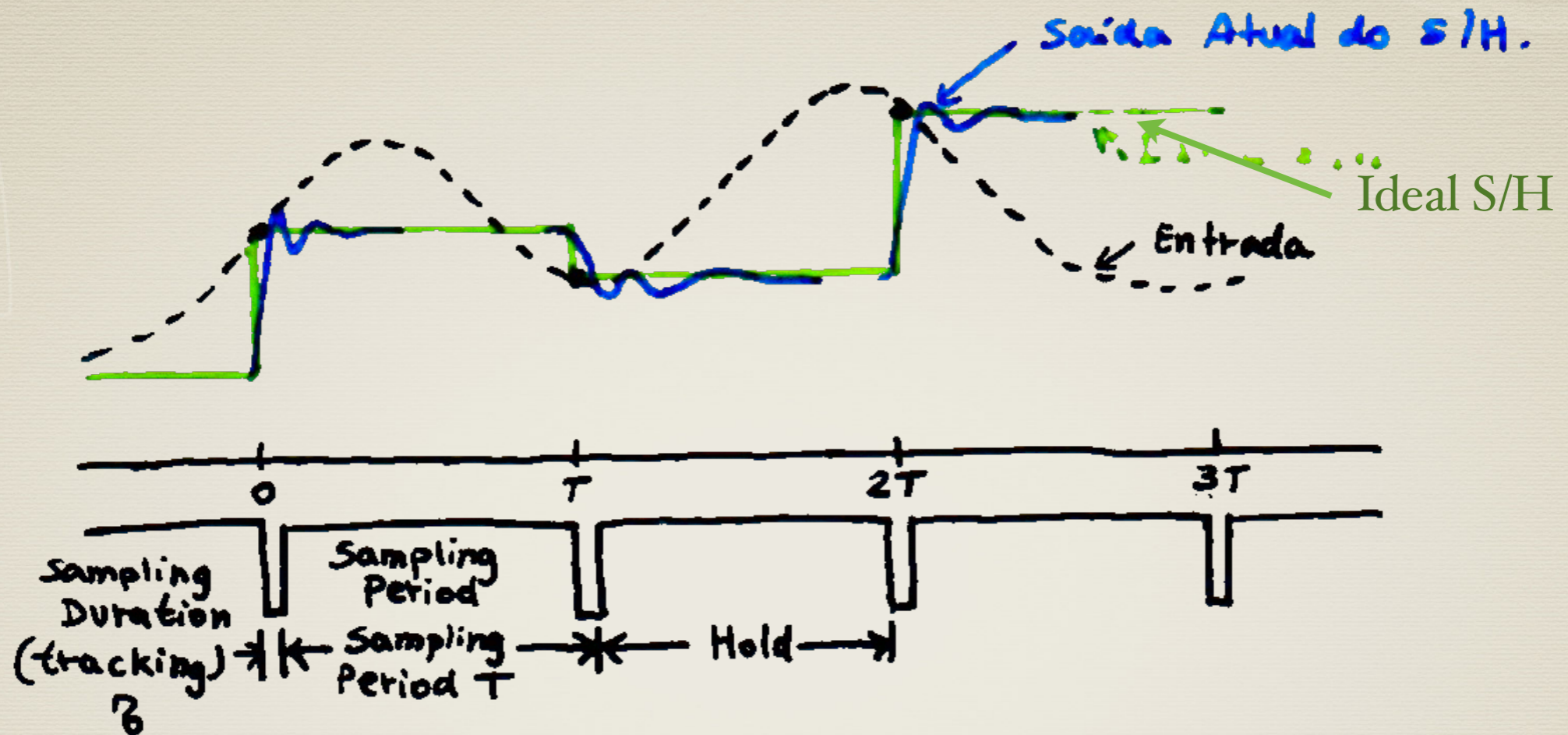


Fig. 2.10 - Entradas e saídas de um dispositivo de S/H com período de amostragem uniforme.

Resposta em Frequência de um ZOH

$$G_{ho}(s) = \frac{1 - e^{-Ts}}{s}$$

ou

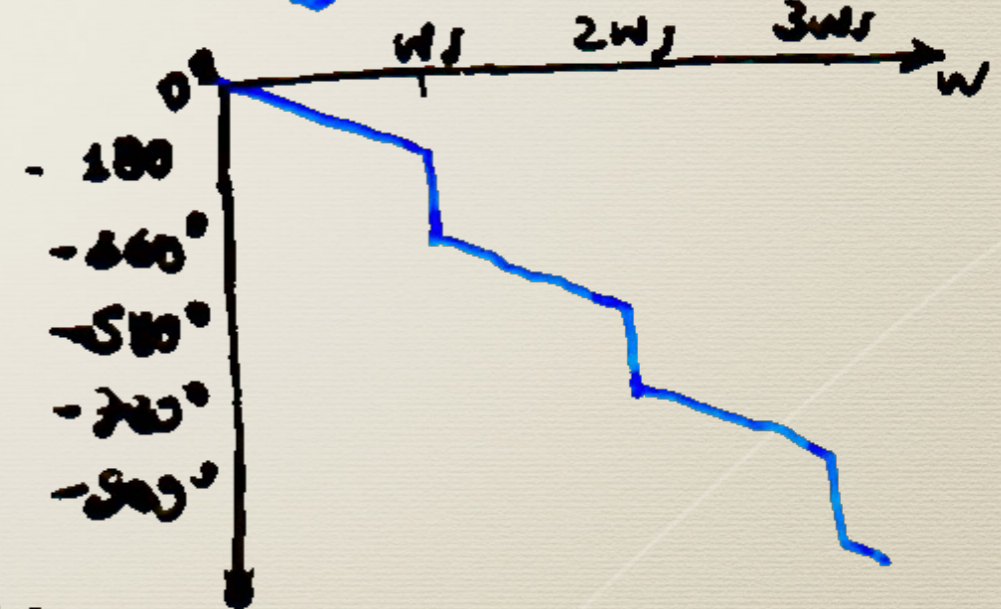
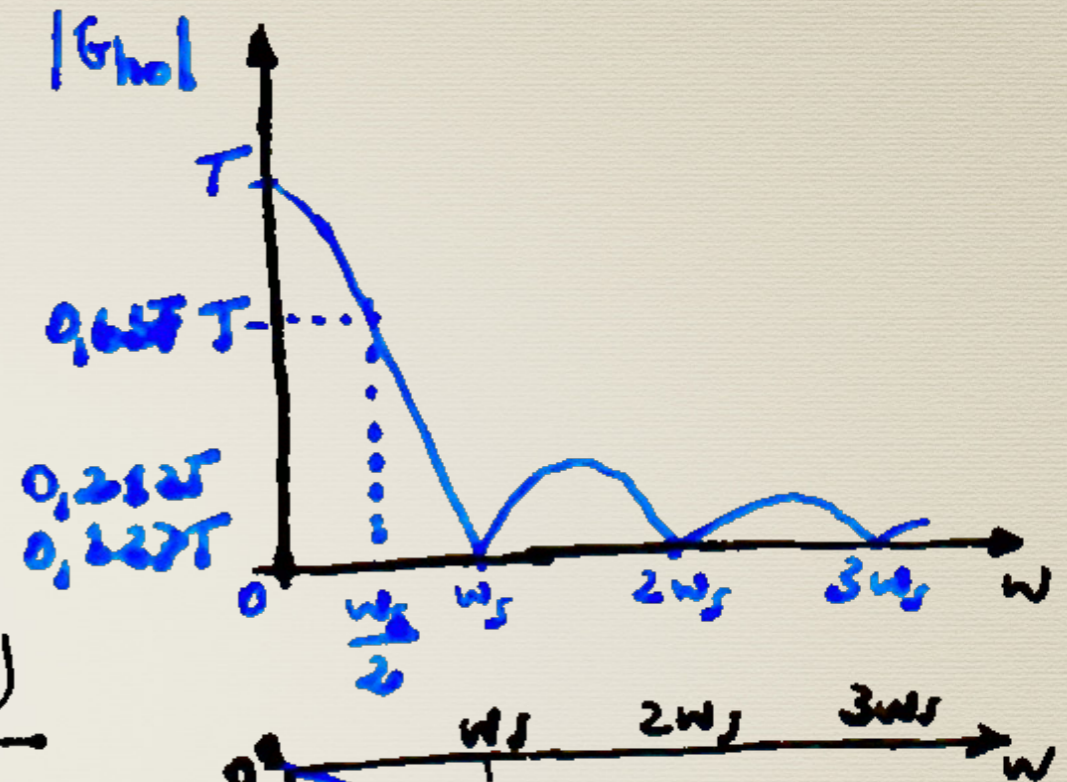
$$G_{ho}(j\omega) = \frac{1 - e^{-Tj\omega}}{j\omega}$$

$$= \frac{j\omega - \frac{1}{2}Tj\omega (e^{\frac{1}{2}Tj\omega} - e^{-\frac{1}{2}Tj\omega})}{2e^{-\frac{1}{2}Tj\omega}}$$

usando relações de Euler:

$$= T \cdot \frac{\sin(\omega T/2)}{\omega T/2} \cdot e^{-\frac{1}{2}Tj\omega}$$

$$|G_{ho}(j\omega)| = T \cdot \frac{\sin(\omega T/2)}{\omega T/2}$$



$\angle G_{ho}$

Resposta em Frequência de um Filtro Ideal

Note que a transformada de Fourier do filtro ideal se assemelha à do sustentador de ordem zero:

$$G_1(j\omega) = \begin{cases} 1, & -\frac{1}{2}\omega_s \leq \omega \leq \frac{1}{2}\omega_s \\ 0, & \text{outros casos} \end{cases}$$

[Ogata, p. 93]

a transformada inversa de Fourier dá:

$$g_1(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_1(j\omega) \cdot e^{j\omega t} \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} e^{j\omega t} \cdot d\omega$$

$$= \frac{1}{2\pi \cdot jt} \left(e^{\frac{1}{2}j\omega_s t} - e^{-\frac{1}{2}j\omega_s t} \right)$$

$$= \frac{1}{\pi t} \cdot \sin \frac{\omega_s t}{2}$$

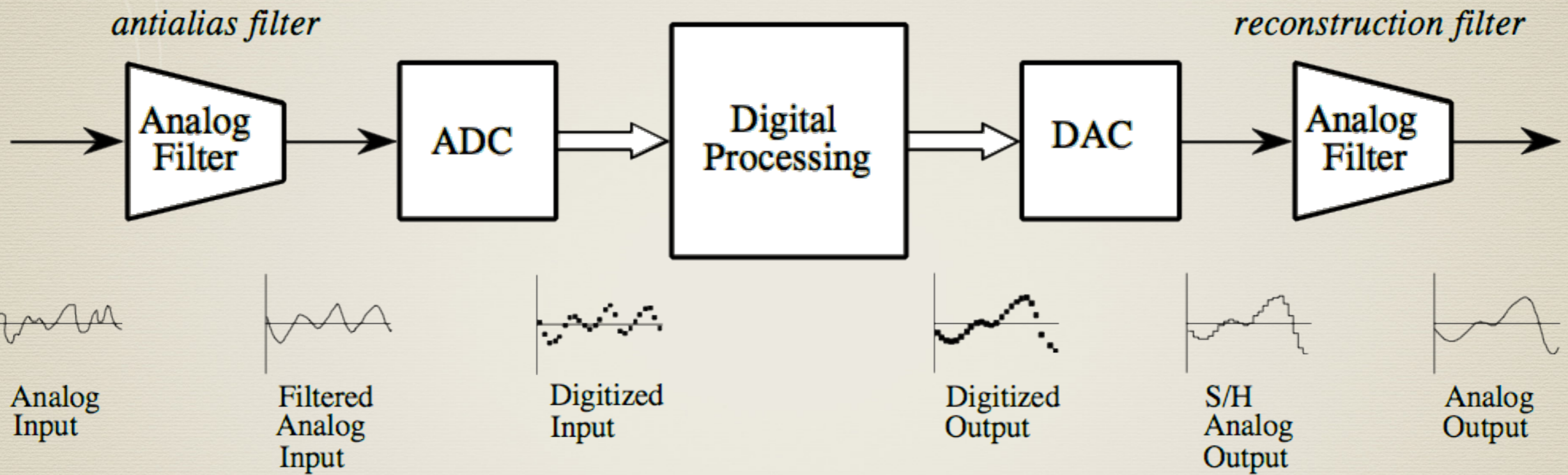


ou

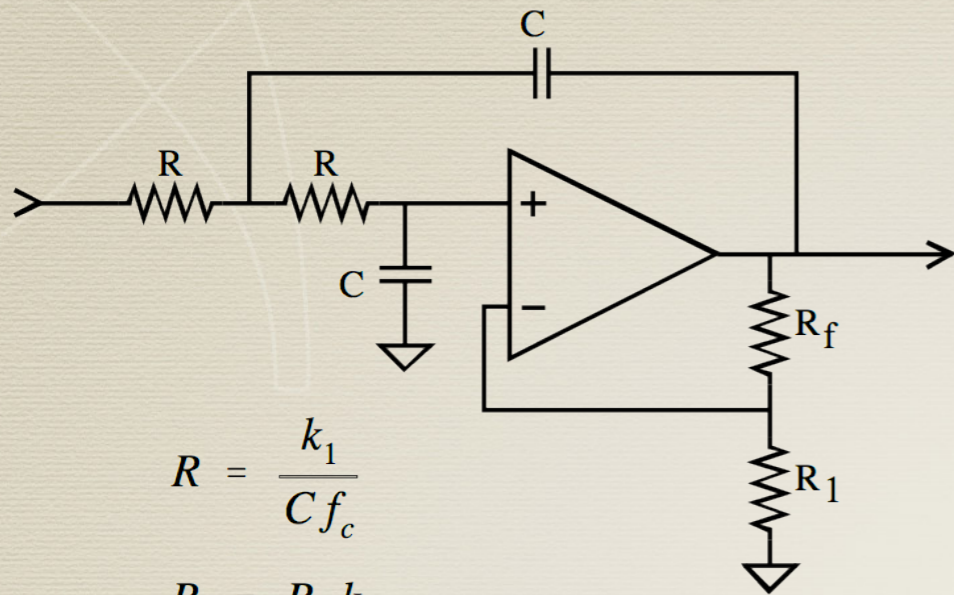
$$g_2(t) = \frac{1}{T} \cdot \frac{\sin(\omega_s t/2)}{\omega_s t/2} \quad \therefore \text{resposta ao impulso de um Filtro PB ideal.}$$

Note que a resposta se estende de $t = -\infty$ até $t = +\infty$. Isto é, existe (!?) uma resposta para $t < 0$ para um impulso aplicado em $t = 0$. Obviamente isto não é realizável no mundo físico real.

Filtros Analógicos



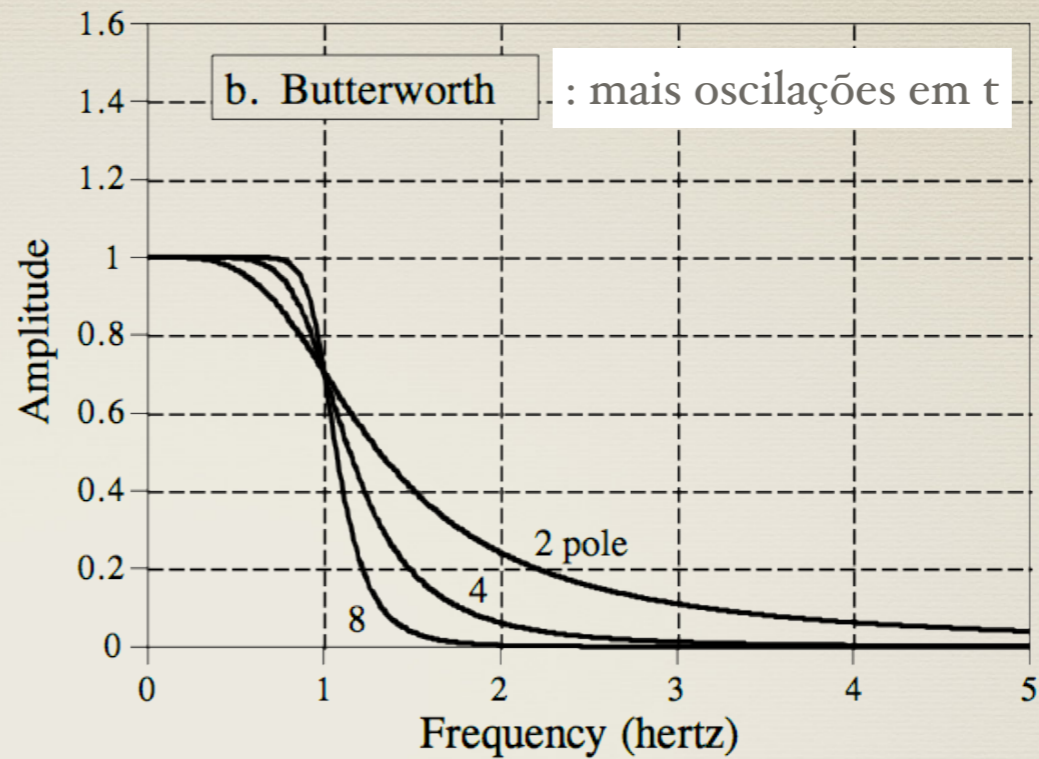
Filtros Analógicos



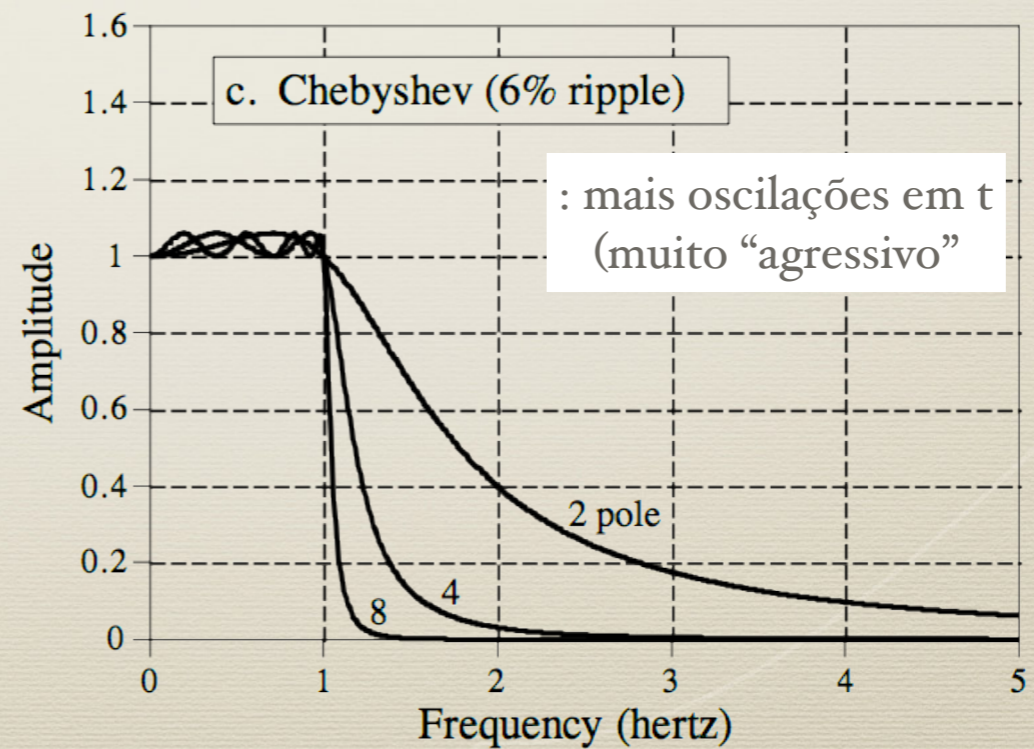
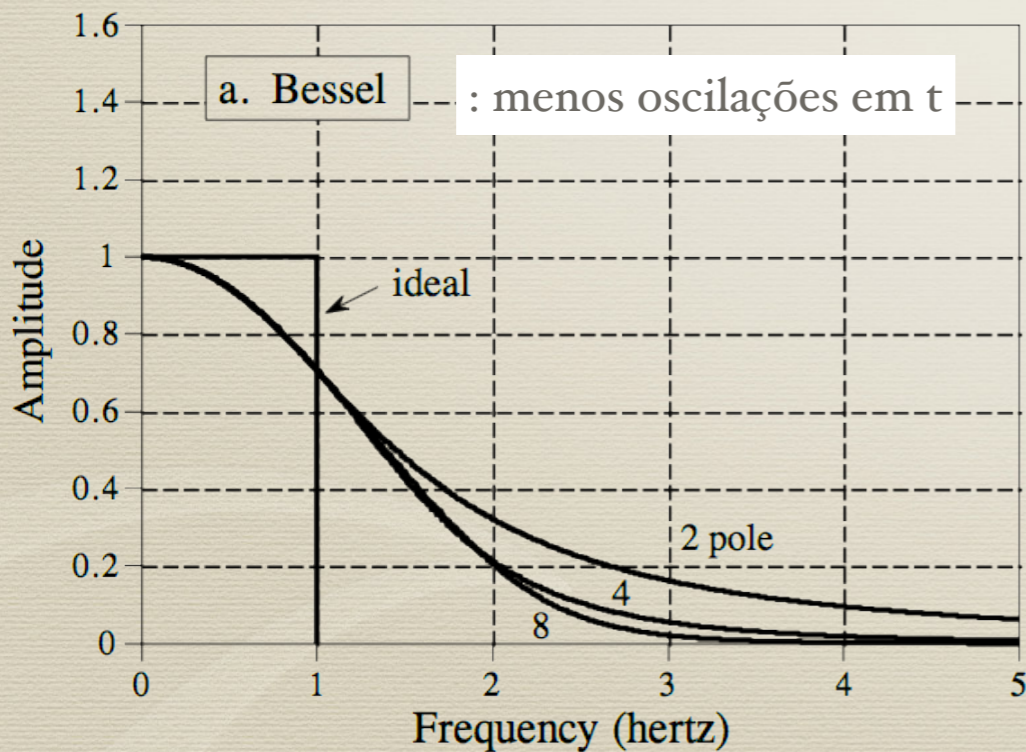
$$R = \frac{k_1}{C f_c}$$

$$R_f = R_1 k_2$$

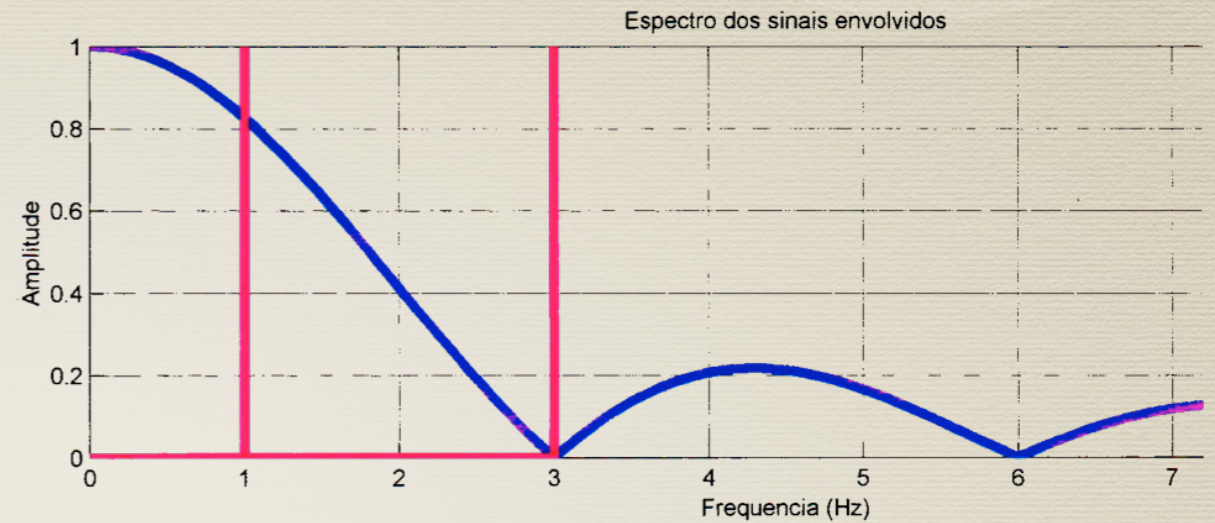
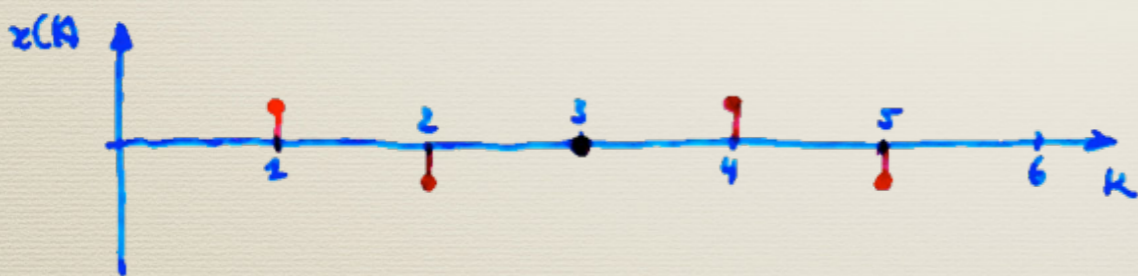
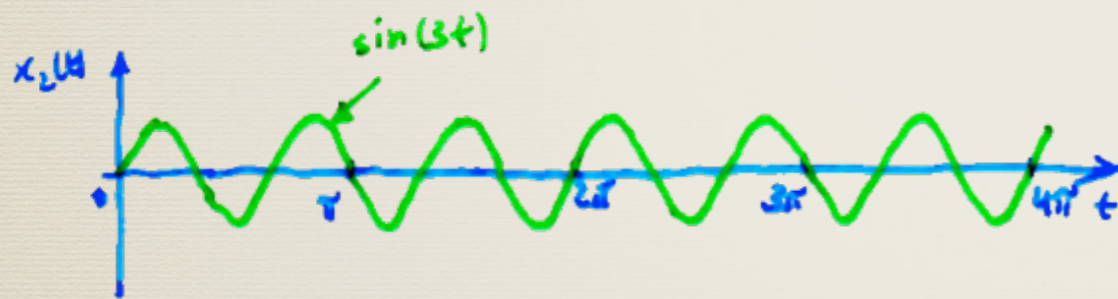
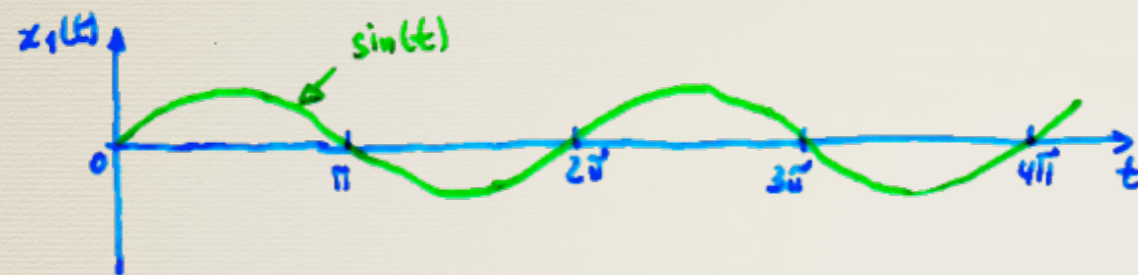
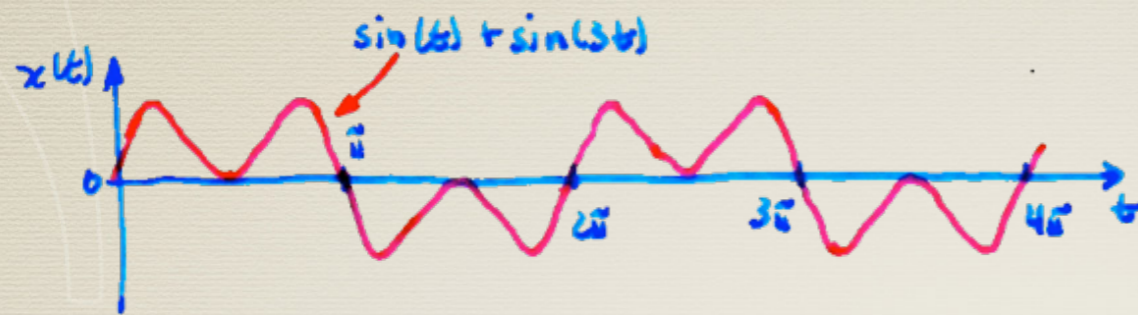
Filtro passa-baixa,
estrutura Sallen-Key,
de 2ª-ordem.



Obs.: Escalas (freq.) lineares!



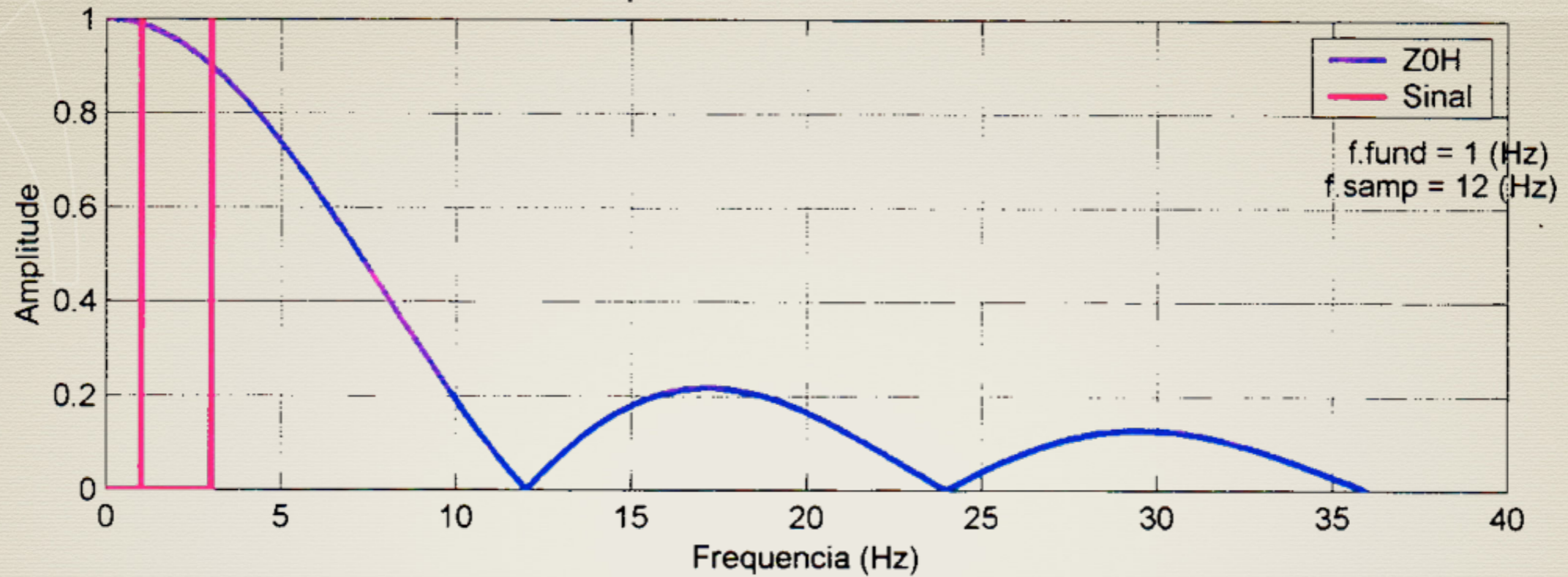
Simulações:



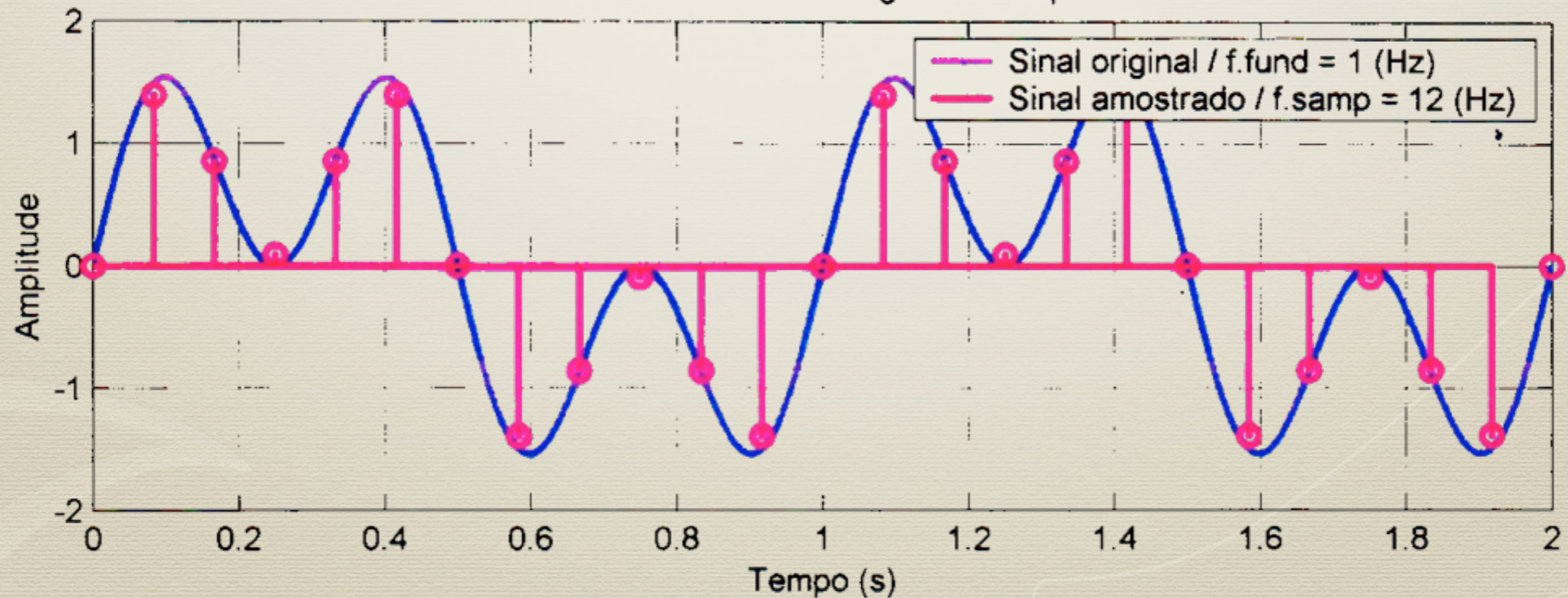
gráficos de $x(t) = \sin(t) + 3\sin(3t)$; $x_1(t) = \sin(t)$; $x_2(t) = \sin(3t)$,
 sinal Amostrado: $x(k)$, onde $\omega_s = 3 \text{ rad/s}$.
 Note que não aparece a frequência: $\omega = 3 \text{ rad/s}$.

Simulações:

Espectro dos sinais envolvidos

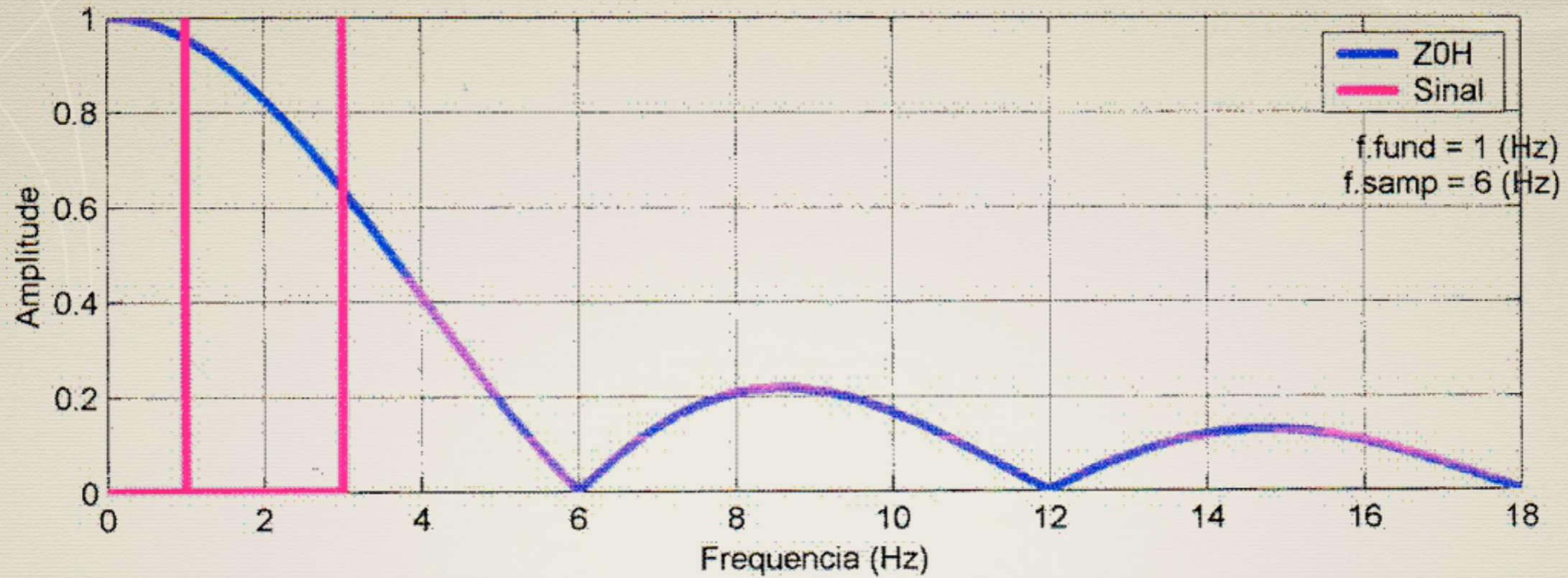


Resultado da amostragem no tempo

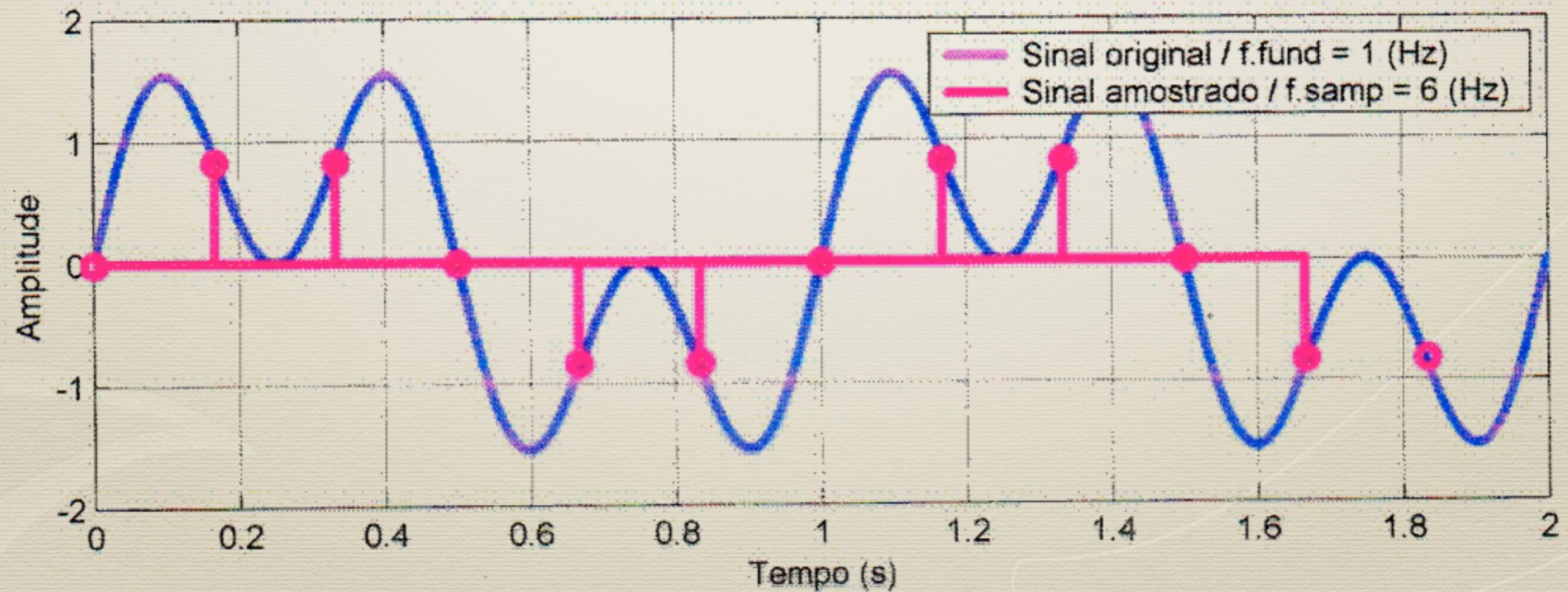


Simulações:

Espectro dos sinais envolvidos

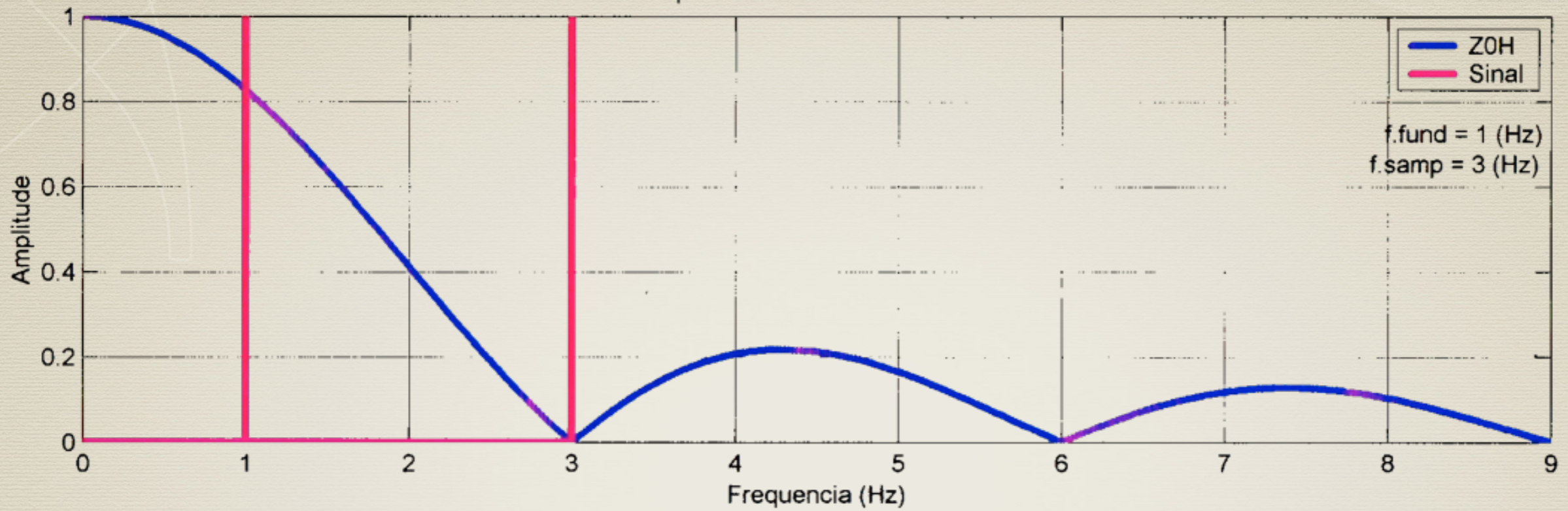


Resultado da amostragem no tempo

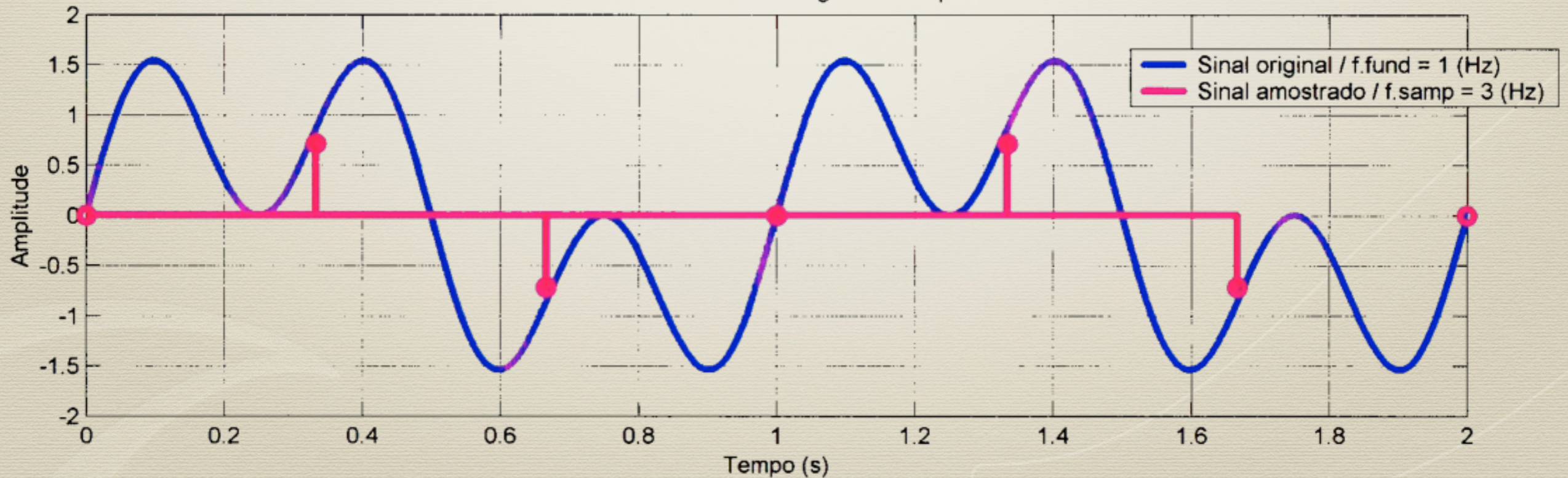


Simulações:

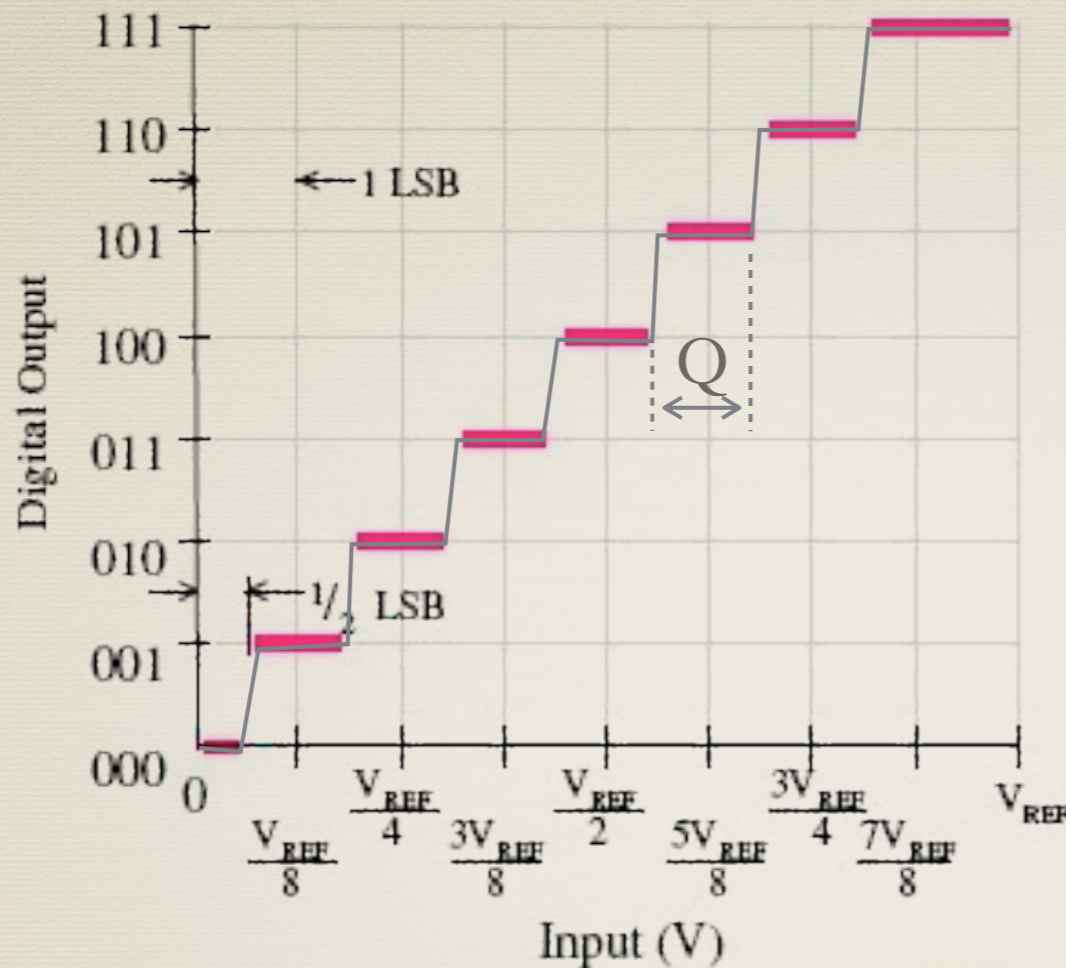
Espectro dos sinais envolvidos



Resultado da amostragem no tempo



Erro de Quantização



Erro de Quantização, Q :

$$Q = \frac{FSR}{2^n}$$

onde:

FSR = Full Scale Range (maior faixa de entrada);

n=No. de bits.

No caso:

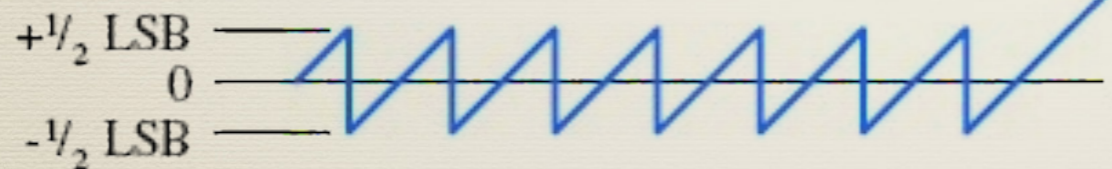
$V_{REF}=10$ Volts,

$$Q = \frac{10}{2^3} = 1,25 V$$

Erro na amostragem:

$$\pm 0,625 V (\pm 0,195\%)$$

ERROR



Obs.: As maiores fontes de erro num sistema de aquisição digital se concentram no circuito de entrada que condiciona o sinal (filtro), limita sua escala (escala) e no circuito de sample-and-hold. Maiores que os comparados ao erro inerente à quantização do sinal de entrada.