

3ª Parte Root Locus:
Respostas de Sistemas Lineares
x plano-s

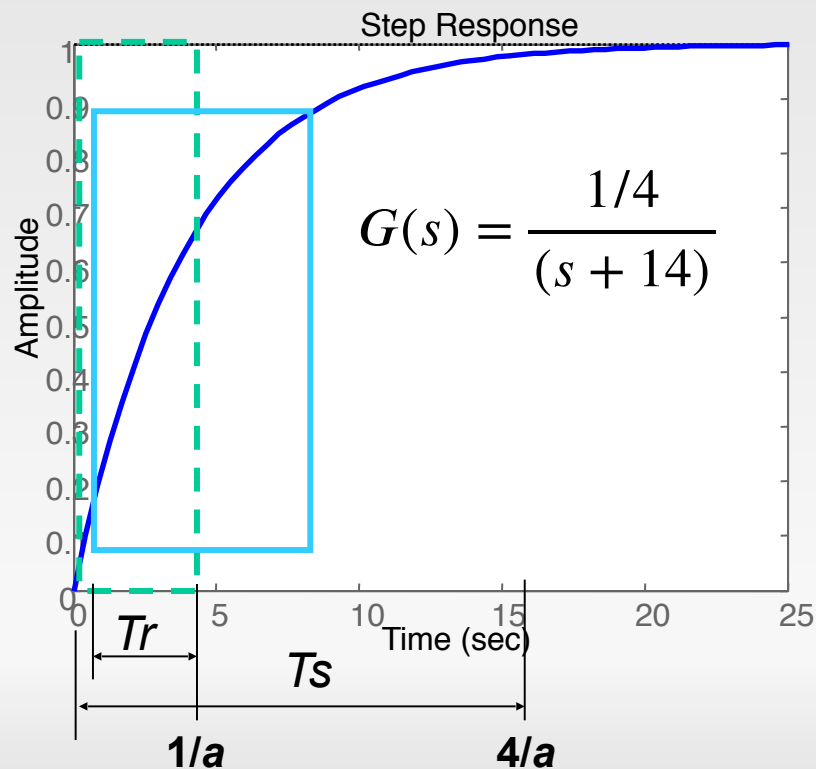
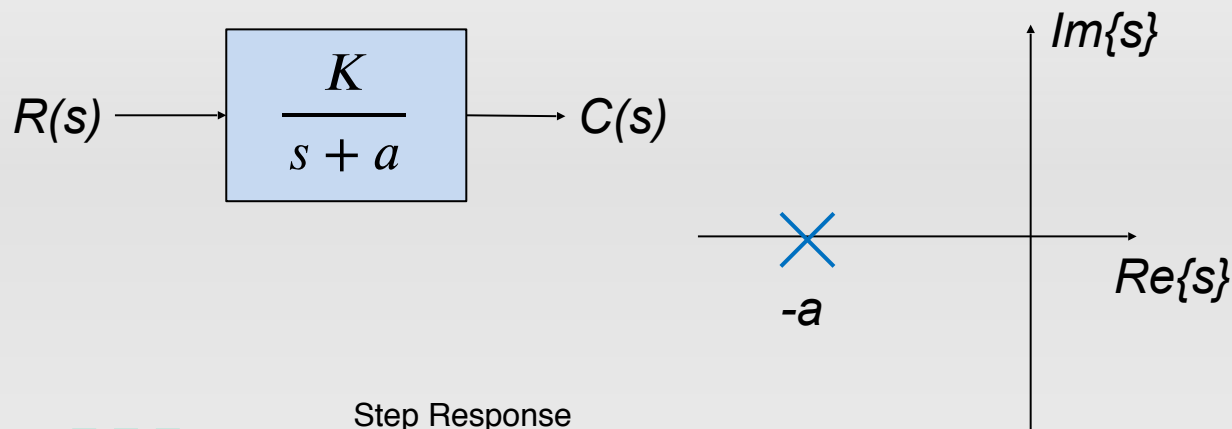


Procedimento Para Projetos

- 1) Desenhe o diagrama do lugar das raízes para o sistema;
- 2) Assuma que é um sistema de 2ª ordem com ou sem zeros e encontre o ganho que cumpra com as especificações de resposta transitória;
- 3) Justifique a aproximação para um sistema de 2ª ordem encontrando a posição de todos os polos mais rápidos e considerando o fato de que estão muito mais afastados do eixo $j\omega$ que os pares dominantes de 2ª-orden!
- 4) Também verifique que os zeros de laço fechado são aproximadamente cancelados pelos polos de maior ordem (os dominantes).
- 5) Se esta hipótese não pode ser justificada, sua solução terá que ser simulada com o fim de garantir de que se cumpre com a especificação para a resposta transitória.

Respostas transitórias

Sistema de 1ª-ordem



Resposta em malha fechada:

$$C(s) = R(s)G(s)$$

$$C(s) = \frac{1}{s} \cdot \frac{K}{(s+a)}$$

$$C(s) = \frac{K/a}{s} - \frac{K/a}{(s+a)}$$

$$c(t) = \frac{K}{a} - \frac{K}{a}e^{-at}$$

Constante de tempo $\rightarrow \tau = \frac{1}{a}$

$$Tr = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

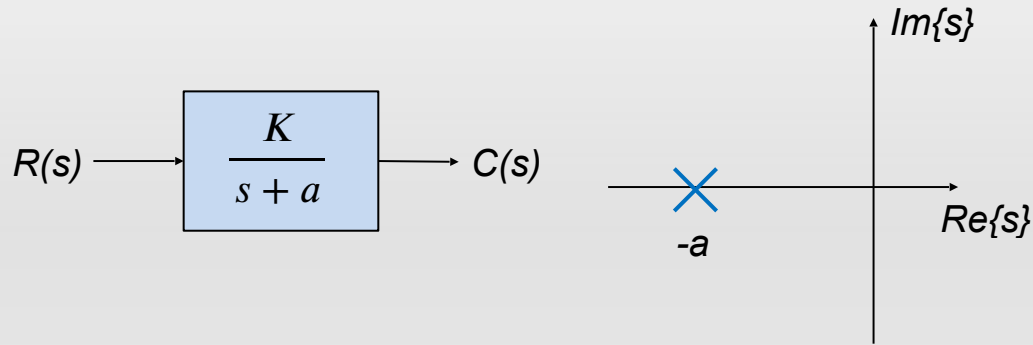
$$Ts = \frac{4}{a}$$

← Tempo de subida:
 $K/a \cdot [0.1 \sim 0.9]$

← Tempo de assentamento:
 $K/a \cdot 0.98$

Respostas transitórias

Sistema de 1ª-ordem



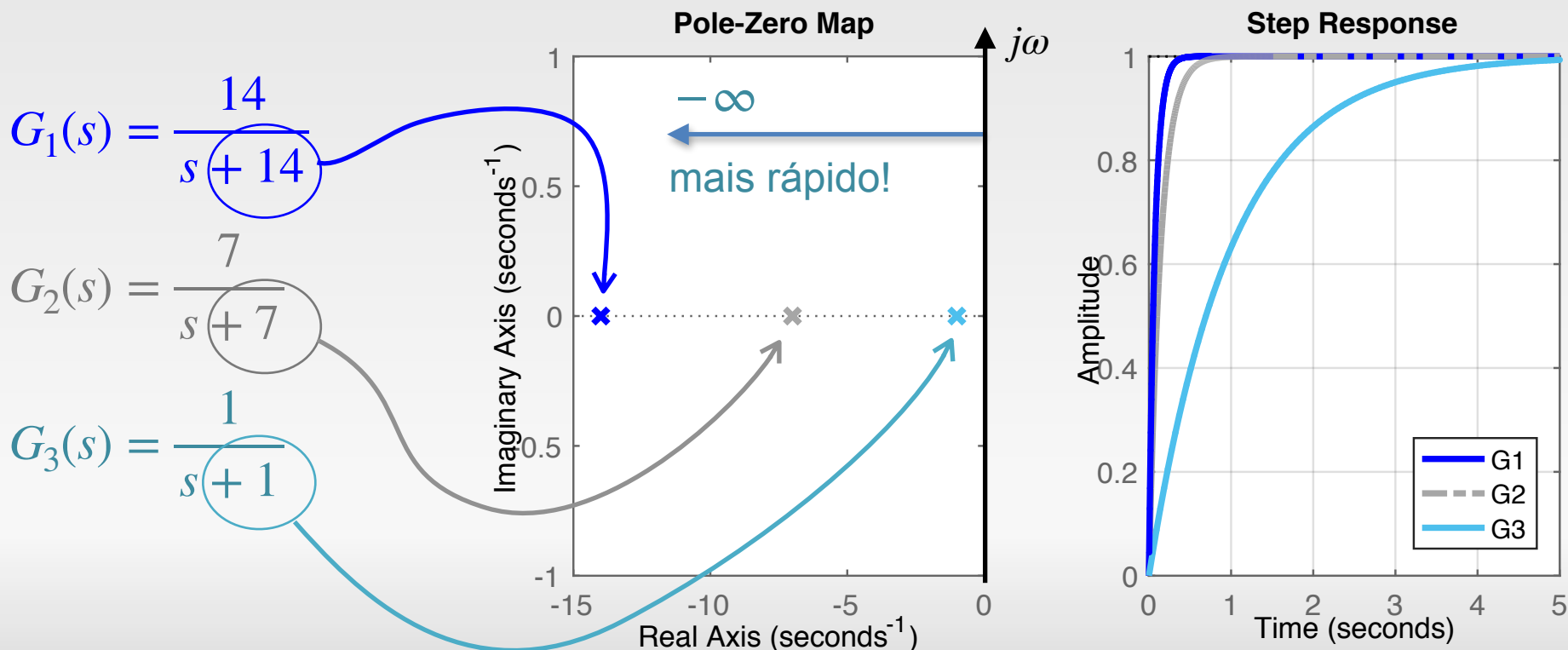
Resposta em malha fechada:

$$c(t) = \frac{K}{a} - \frac{K}{a} e^{-at}$$

Constante de tempo $\rightarrow \tau = \frac{1}{a}$

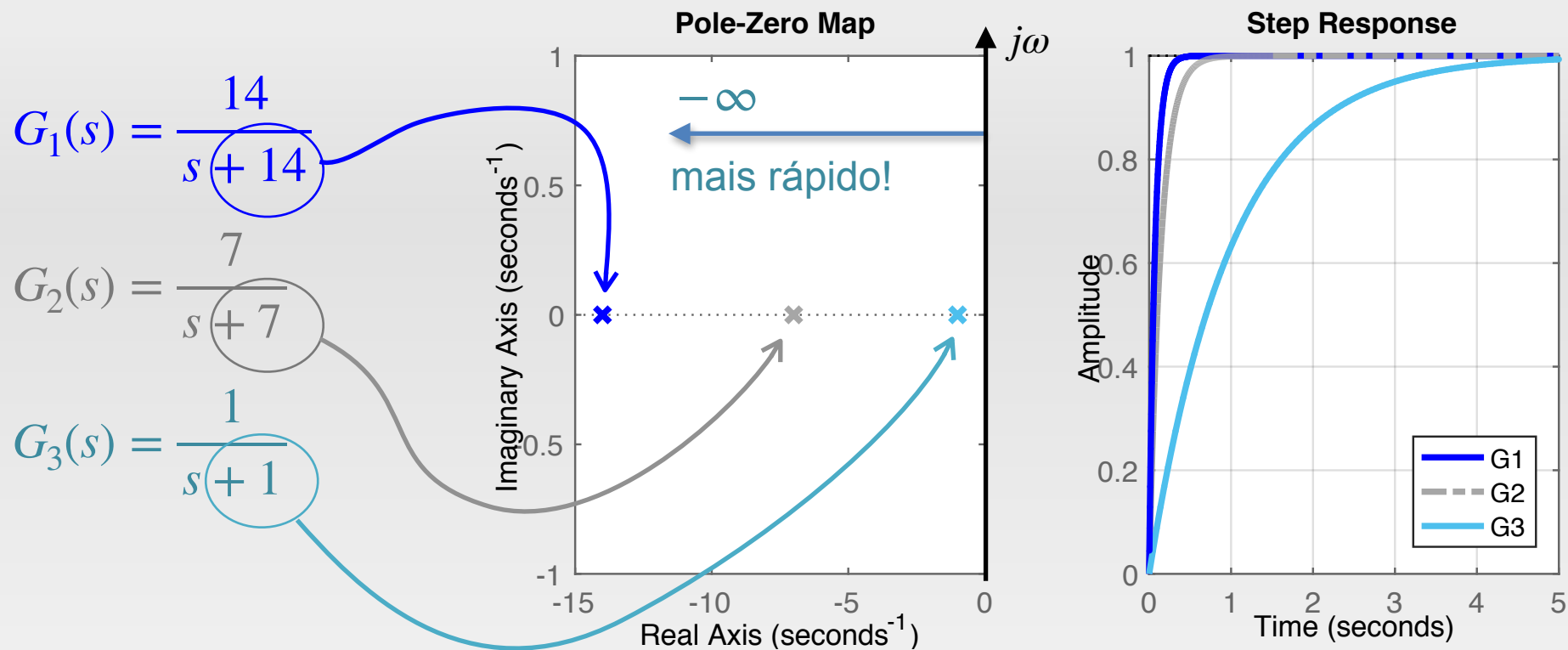
Tempo de assentamento $\rightarrow t_s = \frac{4}{a} = 4\tau$
 (= $K/a \cdot 0,98$)

Comparando 3 sistemas de pólos simples reais:



Respostas transitórias

Sistema de 1ª-ordem



Matlab:

```
>> G1=tf(14,[1 14])
```

G1 =

$$\frac{14}{s + 14}$$

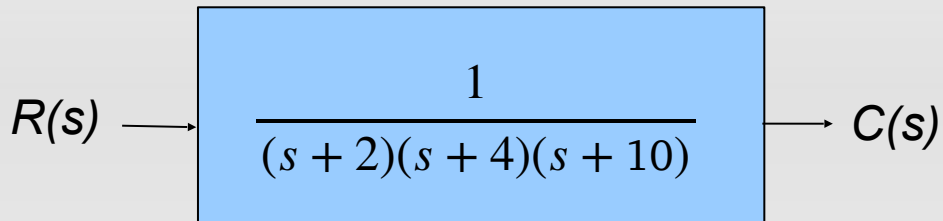
Continuous-time transfer function.

```
>> G2=tf(7,[1 7]);
>> G3=tf(1,[1 1]);
>> figure; subplot(121); pzmap(G1,G2,G3);
>> subplot(122); step(G1,G2,G3);
>> axis([0 5 0 1])
```

Respostas transitorias

Sistema com pólos reais

$$R(s) = \frac{1}{s} \leftarrow \text{Degrau (unitário)}$$



$$C(s) = \frac{1}{s(s+2)(s+4)(s+10)}$$

```
>> num=1;
>> den=poly([0 -2 -4
-10]);
>>
[r,p,k]=residue(num,den)
r =
-0.0021
 0.0208
-0.0313
 0.0125
p =
-10.0000
-4.0000
-2.0000
 0
k =
[]
>>
```

Resposta em malha aberta:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

$$K_1 = \frac{1 \cdot s}{s(s+2)(s+4)(s+10)} \Big|_{s \rightarrow 0} = \frac{1}{80} = 0,0125$$

$$K_2 = \frac{1 \cdot (s+2)}{s(s+2)(s+4)(s+10)} \Big|_{s \rightarrow -2}$$

$$K_2 = \frac{1}{s(s+4)(s+10)} \Big|_{s \rightarrow -2} = -\frac{1}{32} = -0,0312$$

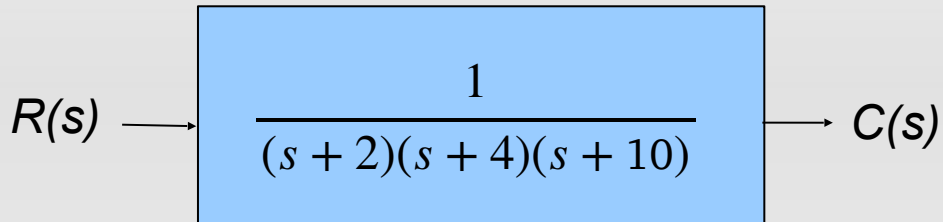
$$K_3 = \frac{1}{s(s+2)(s+10)} \Big|_{s \rightarrow -4} = \frac{1}{48} = 0,0208$$

$$K_4 = \frac{1}{s(s+2)(s+4)} \Big|_{s \rightarrow -10} = -\frac{1}{480} = -0,0021$$

Respostas transitorias

Sistema com pólos reais

$$R(s) = \frac{1}{s} \leftarrow \text{Degrau (unitário)}$$



$$C(s) = \frac{1}{s(s+2)(s+4)(s+10)}$$

Resposta em malha aberta:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

$$K_1 = \frac{1 \cdot s}{s(s+2)(s+4)(s+10)} \Big|_{s \rightarrow 0} = \frac{1}{80} = 0,0125$$

```
>> num=1;
>> den=poly([0 -2 -4
-10]);
>>
[r,p,k]=residue(num,den)
r =
-0.0021
 0.0208
-0.0313
 0.0125
p =
-10.0000
 -4.0000
 -2.0000
  0
k =
 []
>>
```

$[r,p,k] = \text{residue}(b,a)$ finds the residues, poles, and direct term of a **Partial Fraction Expansion** of the ratio of two polynomials, where the expansion is of the form:

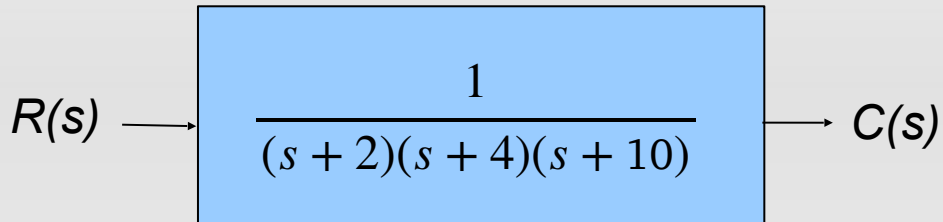
$$\frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = \frac{r_n}{s - p_n} + \dots + \frac{r_2}{s - p_2} + \frac{r_1}{s - p_1} + k(s).$$

$$K_3 = \frac{1}{s(s+2)(s+10)} \Big|_{s \rightarrow -4} = \frac{1}{48} = 0,0208$$

$$K_4 = \frac{1}{s(s+2)(s+4)} \Big|_{s \rightarrow -10} = -\frac{1}{480} = -0,0021$$

Respostas transitorias

Sistema com pólos reais



Resposta em malha aberta:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

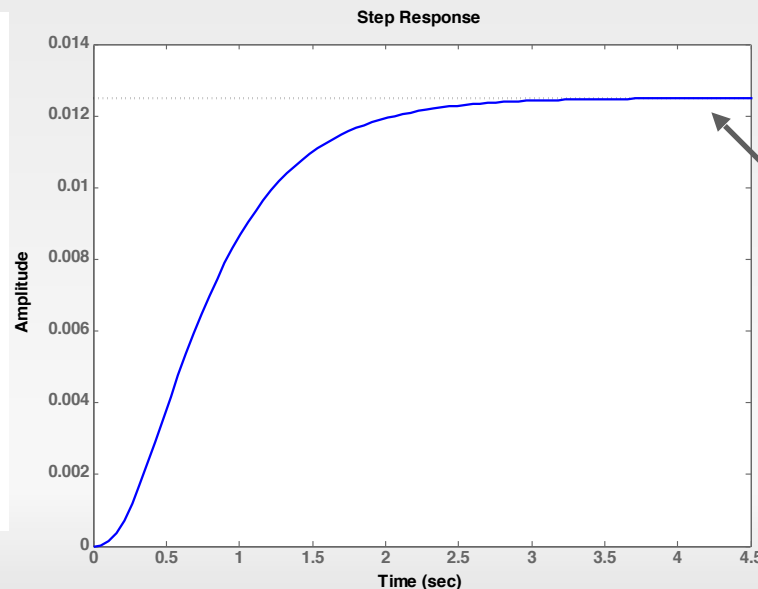
$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

$$c(t) = \frac{1}{80} - \frac{1}{32} e^{-2t} + \frac{1}{48} e^{-4t} - \frac{1}{480} e^{-10t}$$

Matlab:

```
>> num=1;
>> den=poly([-2 -4 -10]);
>> c=tf(num,den);
>> zpk(c)
Zero/pole/gain:
      1
-----
(s+10) (s+4) (s+2)

>> dcgain(c)
ans =    0.0125
>>
```



Valor final:

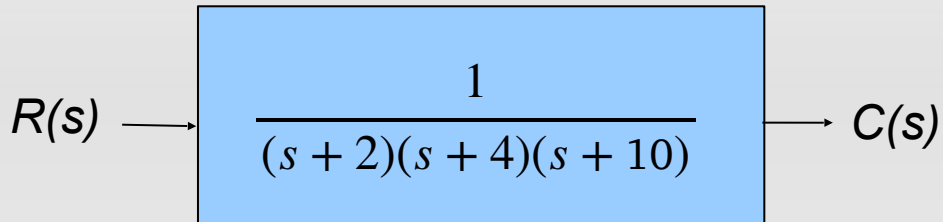
$$c(\infty) = \lim_{s \rightarrow 0} s \cdot C(s)$$

$$c(\infty) = \frac{1 \cdot s}{s(s+2)(s+4)(s+10)}$$

$$c(\infty) = \frac{1}{80} = 0,0125$$

Respostas transitorias

Sistema com pólos reais



Resposta em malha aberta:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

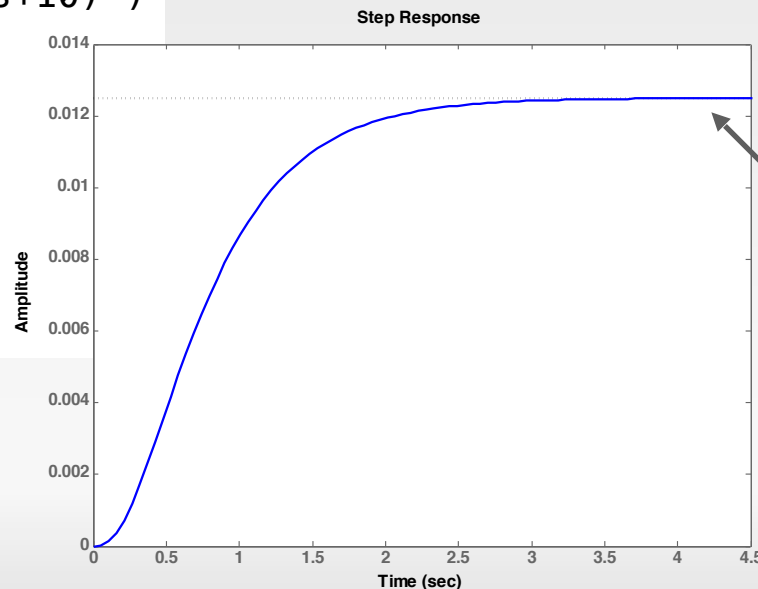
$$c(t) = \frac{1}{80} - \frac{1}{32} e^{-2t} + \frac{1}{48} e^{-4t} - \frac{1}{480} e^{-10t}$$

Matlab: outra forma para $c(\infty)$:

```
>> den=sym(' (s+2)*(s+4)*(s+10) ')
den =
(s+2)*(s+4)*(s+10)

>> subs(den,0)
ans =
80

>>
```



Valor final:

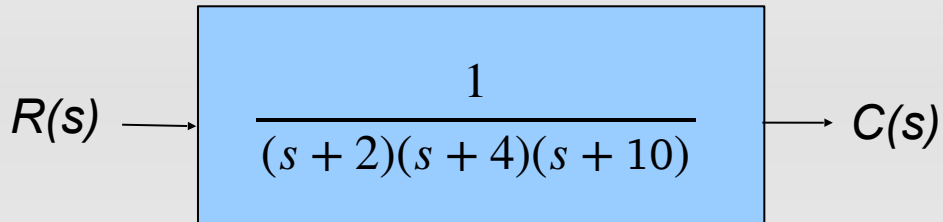
$$c(\infty) = \lim_{s \rightarrow 0} s \cdot C(s)$$

$$c(\infty) = \frac{1 \cdot s}{s(s+2)(s+4)(s+10)}$$

$$c(\infty) = \frac{1}{80} = 0,0125$$

Respostas transitorias

Sistema com pólos reais



Resposta em malha aberta:

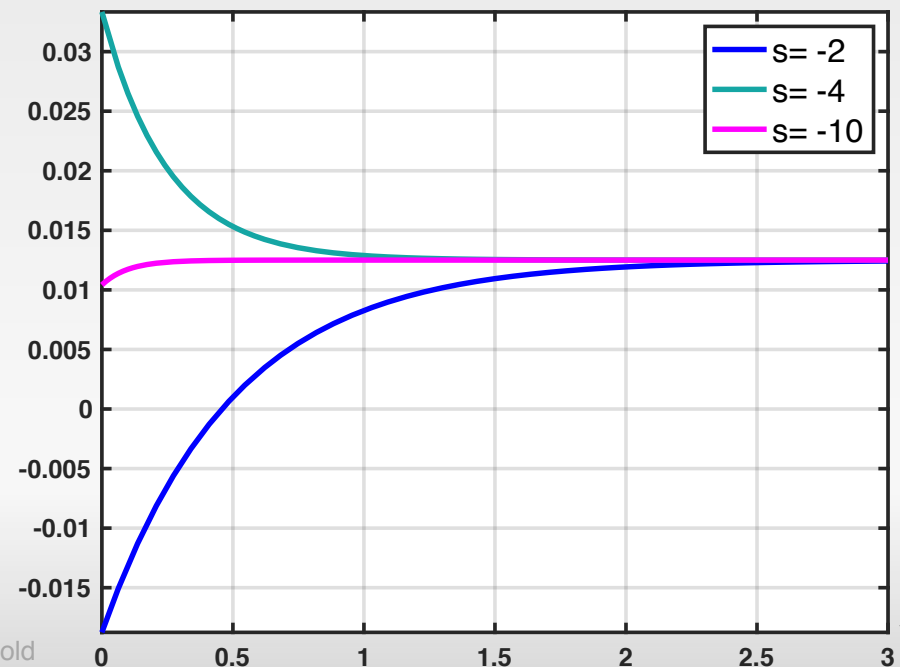
$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

$$c(t) = \frac{1}{80} - \frac{1}{32} e^{-2t} + \frac{1}{48} e^{-4t} - \frac{1}{480} e^{-10t}$$

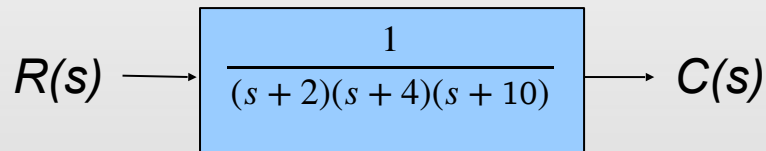
Matlab: verificando impacto de cada polo:

```
>> fplot(@(t)[(1/80)-(1/32)*exp(-2*t)], [0 3])  
>> hold on  
>> fplot(@(t)[(1/80)+(1/48)*exp(-4*t)], [0 3])  
>> hold on  
>> fplot(@(t)[(1/80)-(1/480)*exp(-10*t)], [0 3])  
>> legend('s= -2', 's= -4', 's= -10')
```

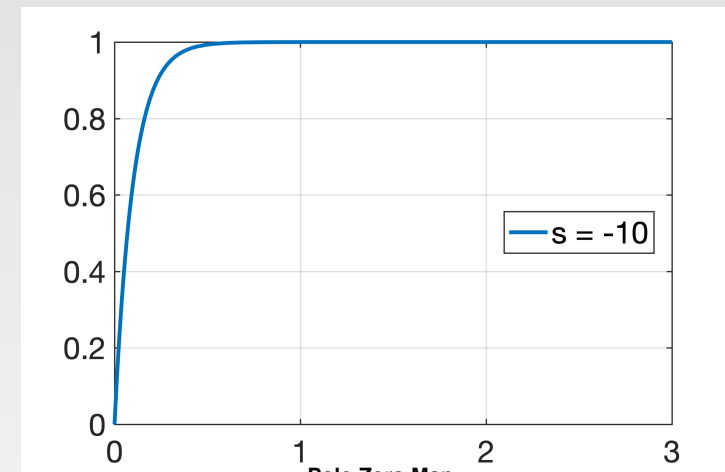
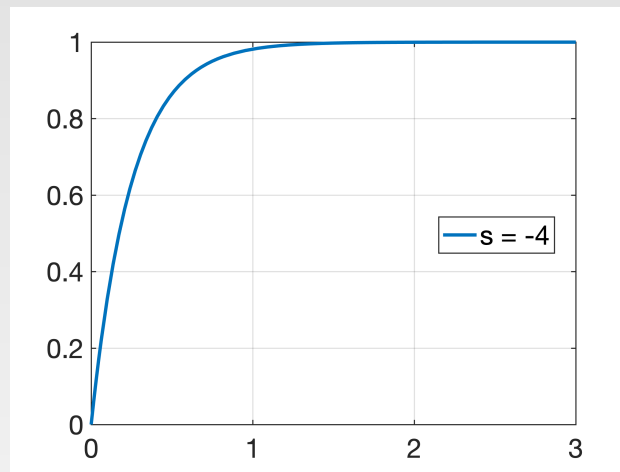
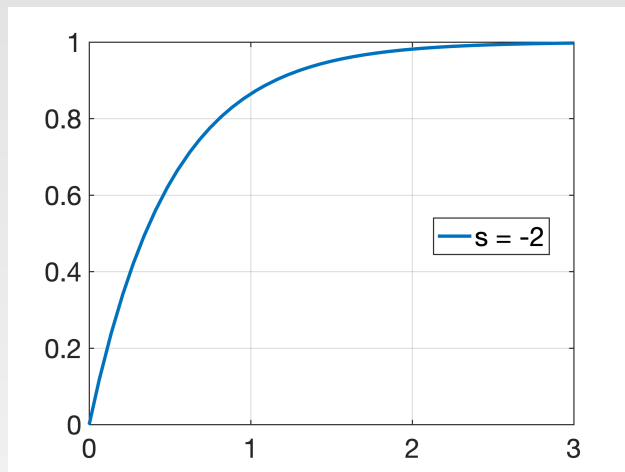


Respostas transitorias

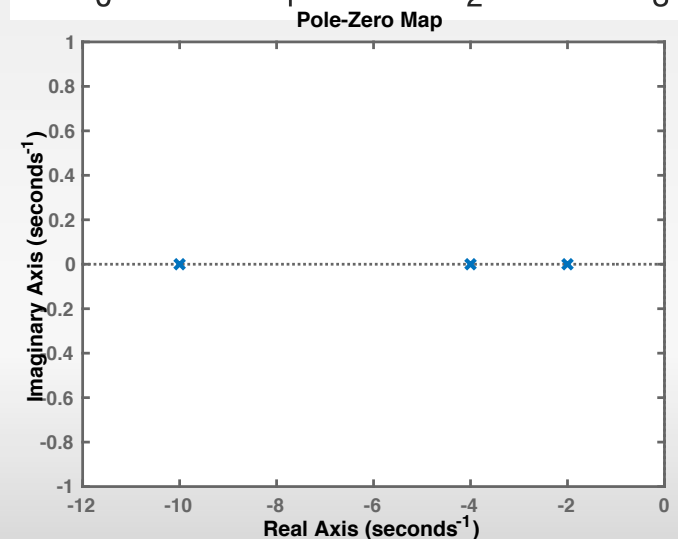
Sistema com pólos reais



Avaliando as respostas de cada pólo individualmente:

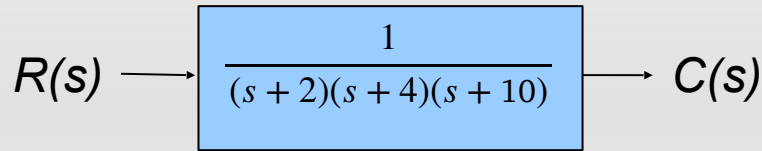


```
>> figure; fplot(@(t)[(1-exp(-2*t))],[0 3])
>> legend('s = -2')
>> figure; fplot(@(t)[(1-exp(-4*t))],[0 3])
>> legend('s = -4')
>> figure; fplot(@(t)[(1-exp(-10*t))],[0 3])
>> legend('s = -10')
>> figure; pzmap(c)
```



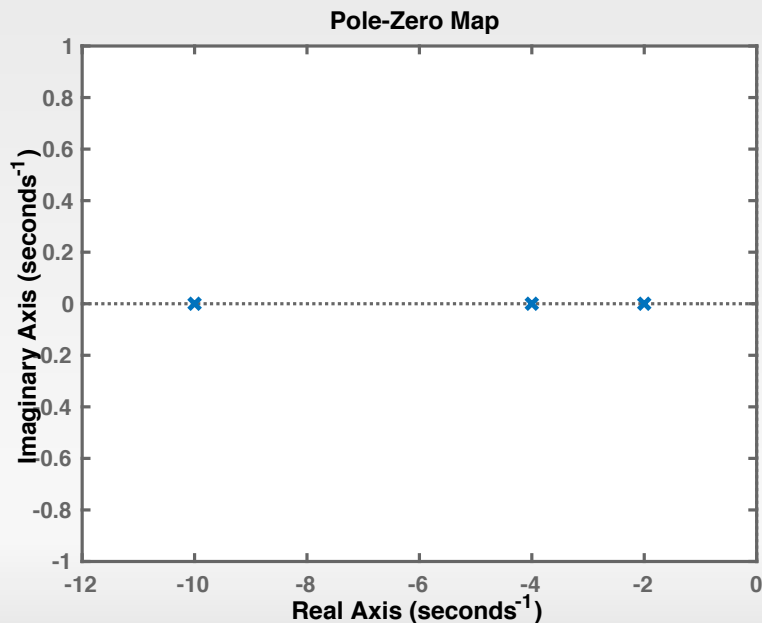
Respostas transitorias

Sistema com pólos reais



Matlab:

```
>> figure(3); pzmap(c)
```

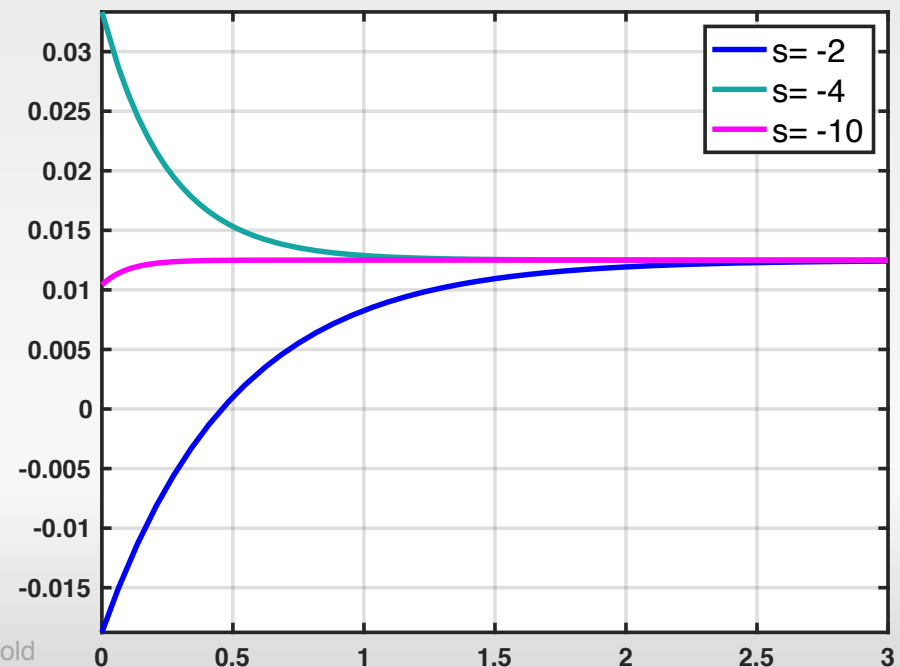


Resposta em malha aberta:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

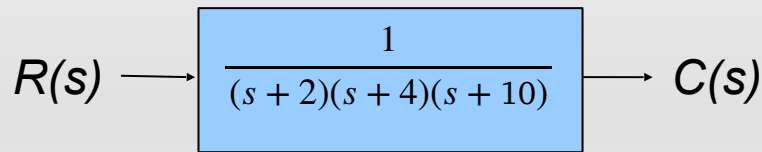
$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

$$c(t) = \frac{1}{80} - \frac{1}{32} e^{-2t} + \frac{1}{48} e^{-4t} - \frac{1}{480} e^{-10t}$$



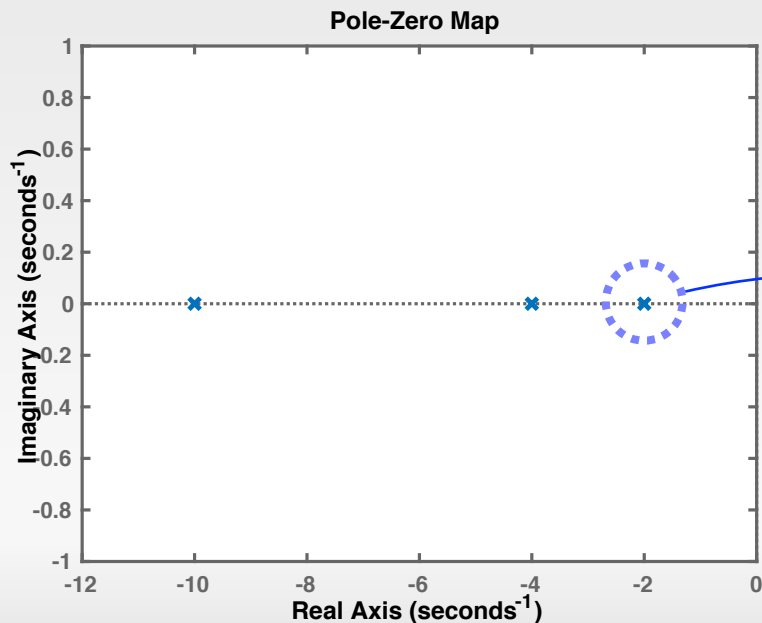
Respostas transitorias

Sistema com pólos reais



Matlab:

```
>> figure(3); pzmap(c)
```

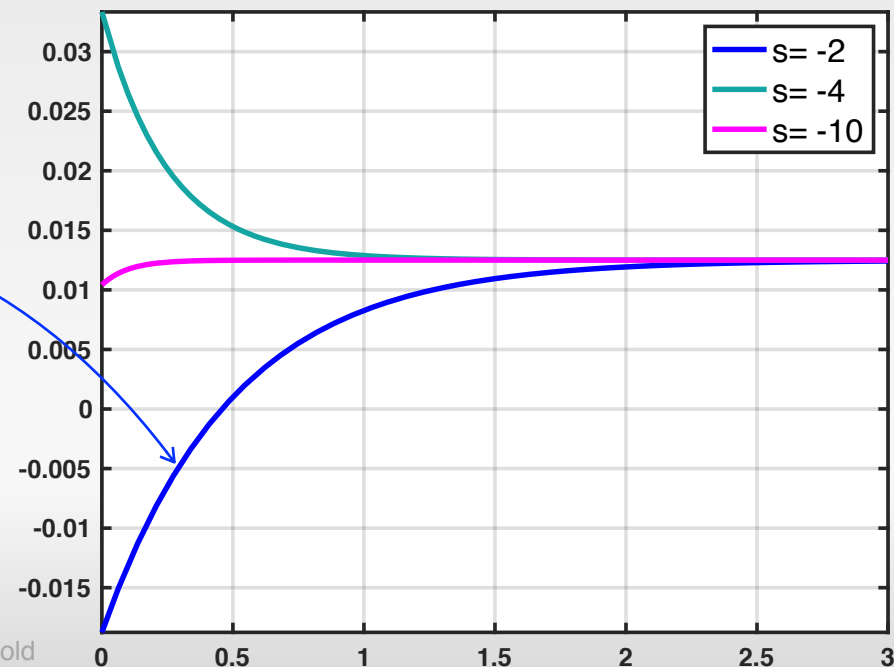


Resposta em malha aberta:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

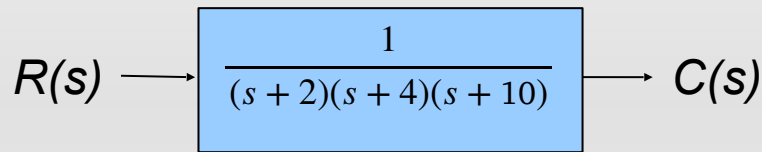
$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

$$c(t) = \frac{1}{80} - \frac{1}{32} e^{-2t} + \frac{1}{48} e^{-4t} - \frac{1}{480} e^{-10t}$$



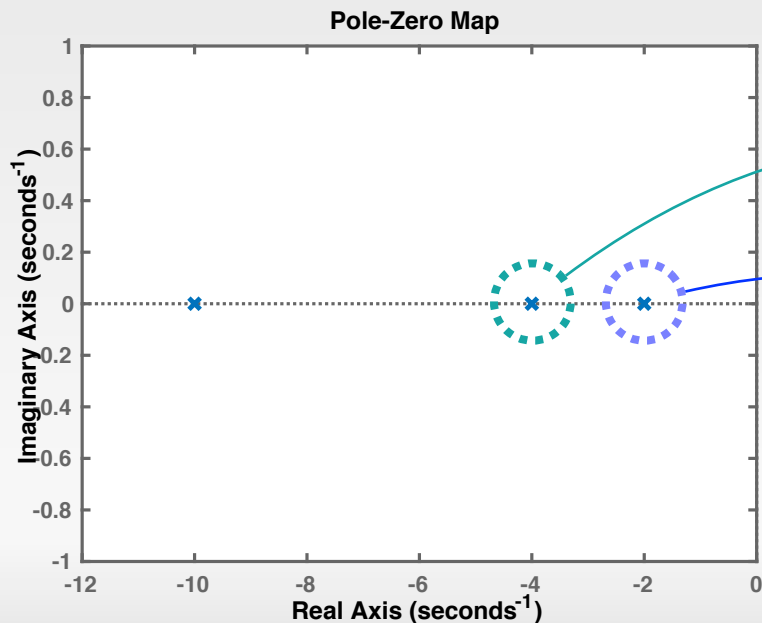
Respostas transitorias

Sistema com pólos reais



Matlab:

```
>> figure(3); pzmap(c)
```

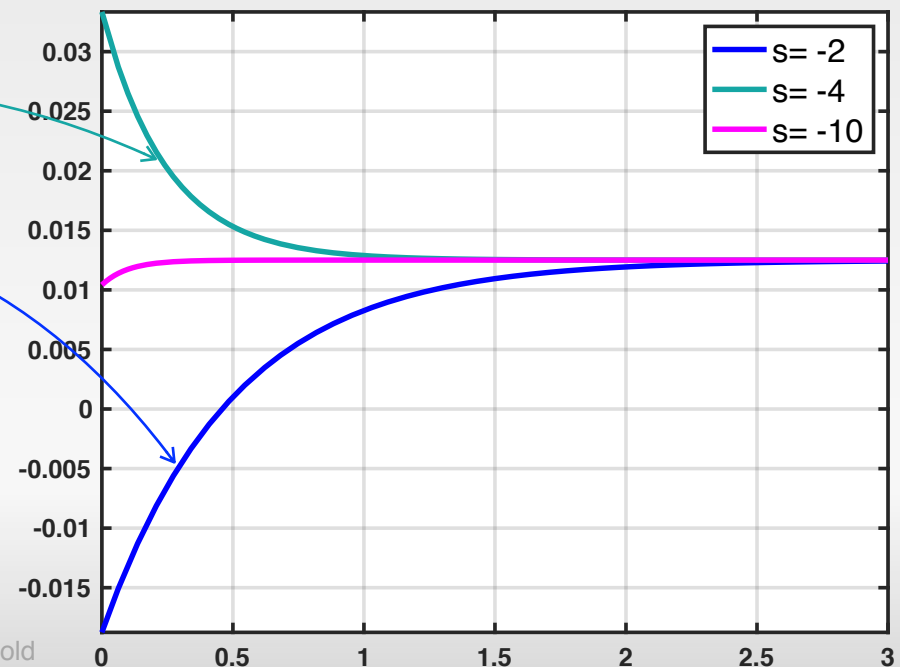


Resposta em malha aberta:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

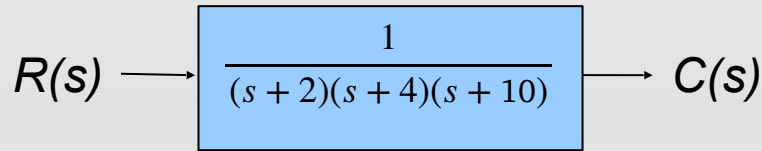
$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

$$c(t) = \frac{1}{80} - \frac{1}{32} e^{-2t} + \frac{1}{48} e^{-4t} - \frac{1}{480} e^{-10t}$$



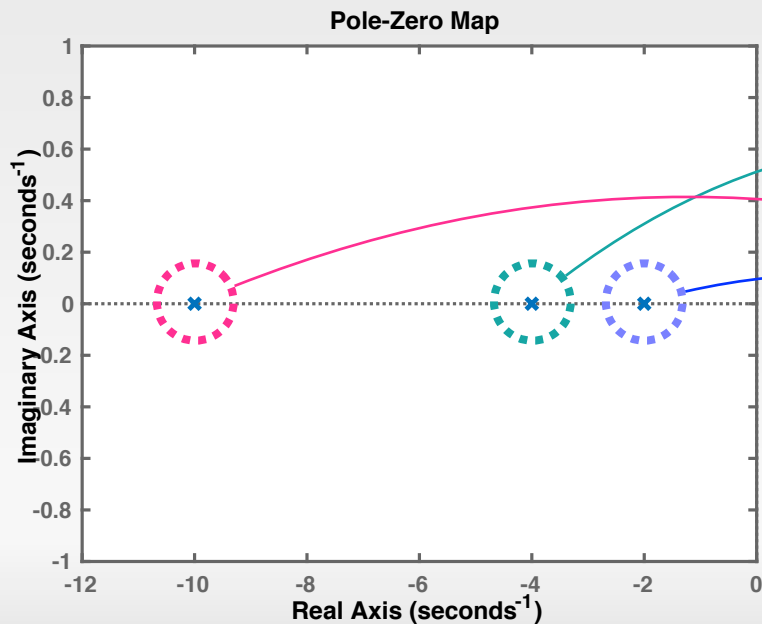
Respostas transitorias

Sistema com pólos reais



Matlab:

```
>> figure(3); pzmap(c)
```

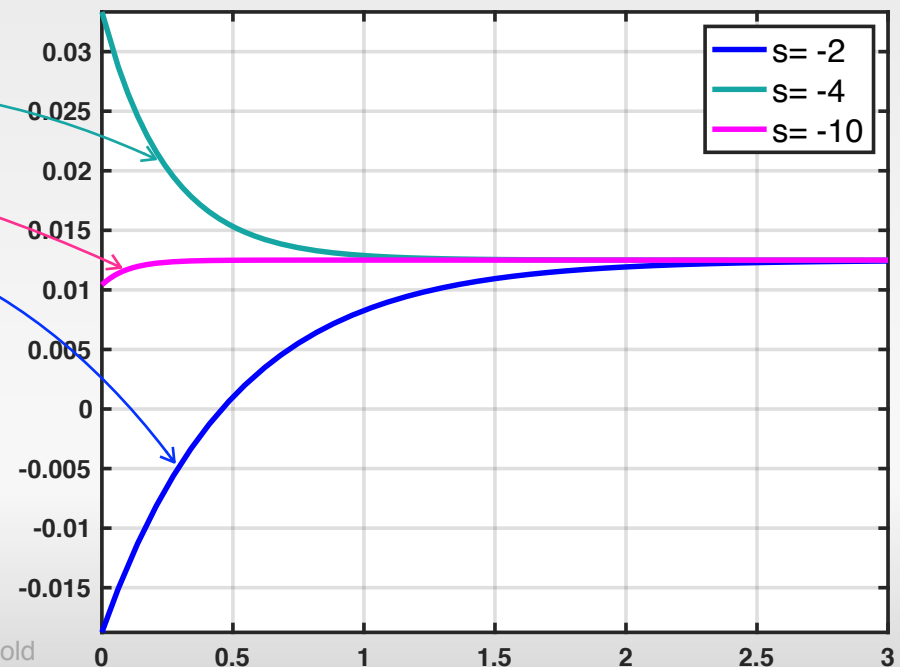


Resposta em malha aberta:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

$$c(t) = \frac{1}{80} - \frac{1}{32} e^{-2t} + \frac{1}{48} e^{-4t} - \frac{1}{480} e^{-10t}$$



Respostas de Sistemas de 2^a ordem



Desenho do RL e encontrando pontos chaves

Exemplo:
$$G(s) = \frac{K(s^2 - 4s + 20)}{(s + 2)(s + 4)}$$

$\zeta = 0.45 \quad \theta = 180^\circ - \arccos(0.45) = 116.7^\circ$

```
>> num=[1 -4 20];  
>> den=conv([1 2],[1 4]);  
>> g=tf(num,den)
```

Transfer function:
 $s^2 - 4s + 20$

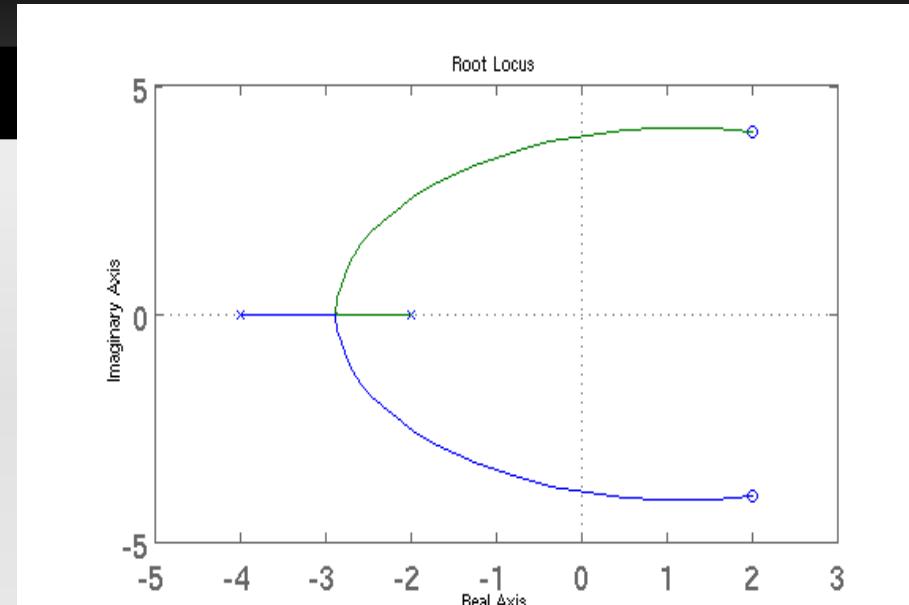
 $s^2 + 6s + 8$

```
>> zpk(g)  
Zero/pole/gain:  
 $(s^2 - 4s + 20)$ 
```

 $(s+4) (s+2)$

```
>> roots(num)  
ans =  
 2.0000 + 4.0000i  
 2.0000 - 4.0000i
```

```
>> rlocus(g)  
>>
```



Questões:

- Cruzamento com eixo $j\omega$
- Ponto de quebra
- Faixa de K (ganho):

Respostas transitorias

Sistemas de 2a-orden:

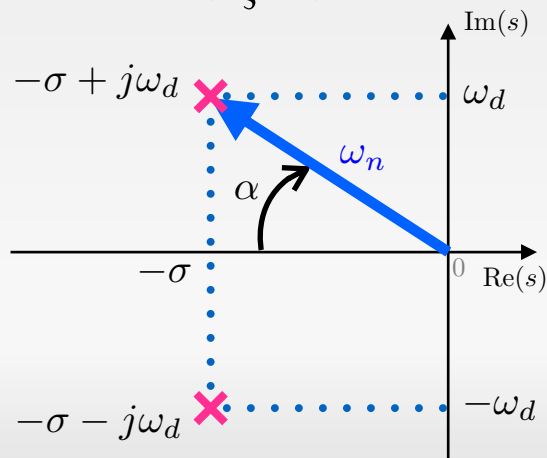
$$R(s) = \frac{1}{s} \rightarrow \boxed{\frac{FTMF(s)}{s^2 + as + b}} \rightarrow Y(s)$$

$$FTMF(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{b}{s^2 + as + b}$$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s + p_1)(s + p_2)} = K \left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right)$$

pólos em: $s = \sigma \pm j\omega_d$ ou: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$

$$0 < \zeta < 1$$



$$\sigma = \omega_n \cos(\alpha) = \omega_n \zeta;$$

$$\omega_d = \omega_n \sin(\alpha) = \omega_n \sqrt{1 - \zeta^2};$$

$$\zeta = \cos(\alpha);$$

$$\sin(\alpha) = \sqrt{1 - \zeta^2};$$

3 casos:

1) Raízes reais:

a) Distintas;

b) Iguais;

2) Raízes complexas:

a) Distante de $j\omega$;

b) Sobre $j\omega$.

Respuestas transitorias

Sistemas de 2a-orden:

$$R(s) = \frac{1}{s} \rightarrow \boxed{\frac{FTMF(s)}{s^2 + as + b}} \rightarrow Y(s)$$

3 casos:

1) Raíces reais:

- a) Distintas;
- b) Iguais

2) Raíces complexas:

- a) Distante de $j\omega$;
- b) Sobre $j\omega$.

$$G_{1a}(s) = \frac{9}{s^2 + 9s + 9}$$

2 pólos reais em $-p_1$ e $-p_2$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$$

$$w_n^2 = 9 \rightarrow w_n = 3$$

$$\zeta = \frac{9}{6} = 1,5$$

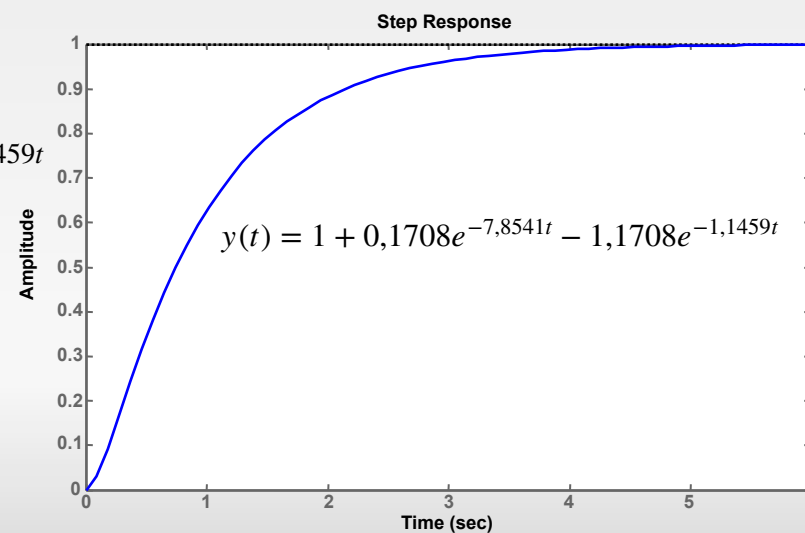
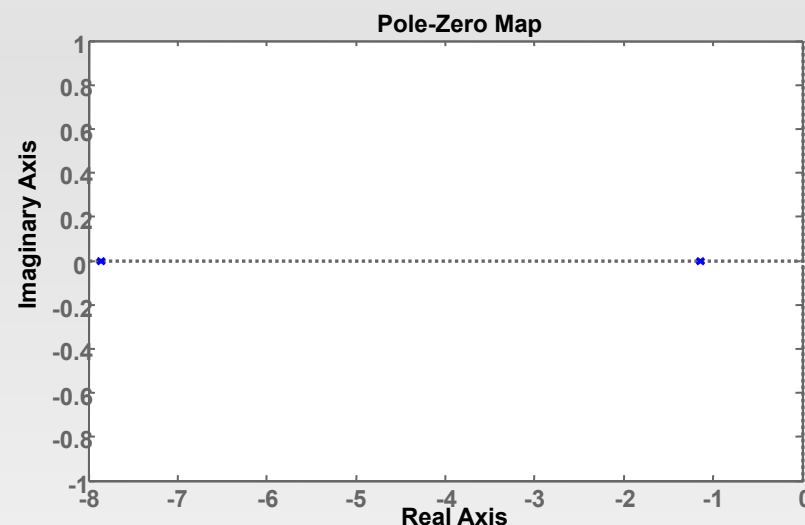
$$y(t) = K_1 e^{-p_1 t} + K_2 e^{-p_2 t}$$

$$y(t) = 1 + 0,1708e^{-7,8541t} - 1,1708e^{-1,1459t}$$

Sistema superamortecido \Rightarrow
($\zeta > 1$)

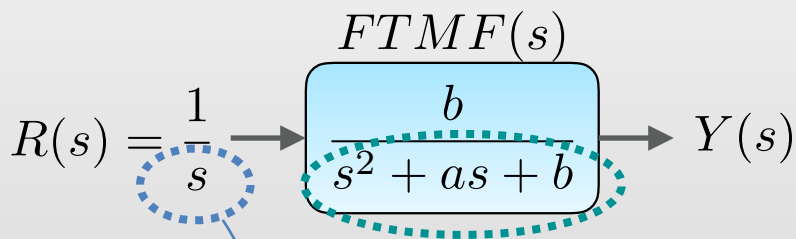
Overdamped response \Rightarrow

Sistema superamortecido \Rightarrow ($\zeta > 1$)



Respuestas transitorias

Sistemas de 2a-orden:



3 casos:

- 1) Raíces reais:
 - a) Distintas;
 - b) Iguais

$$G_{1a}(s) = \frac{9}{s^2 + 9s + 9}$$

2 pólos reais em $-p_1$ e $-p_2$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$$

$$w_n^2 = 9 \rightarrow w_n = 3$$

$$\zeta = \frac{9}{6} = 1,5$$

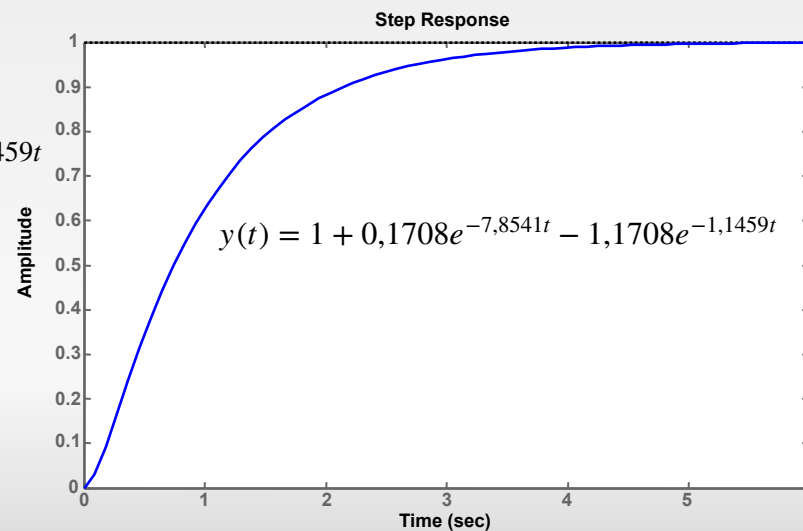
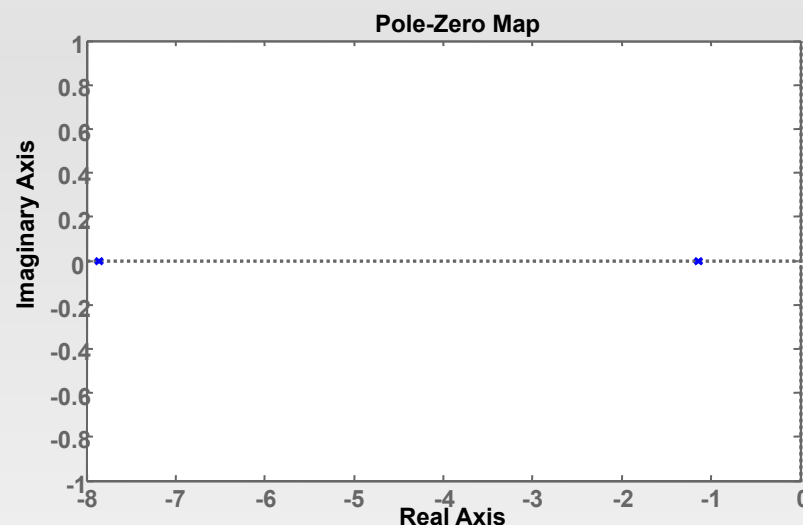
$$y(t) = K_1 e^{-p_1 t} + K_2 e^{-p_2 t}$$

$$y(t) = 1 + 0,1708e^{-7,8541t} - 1,1708e^{-1,1459t}$$

Sistema **superamortecido** \Rightarrow
 $(\zeta > 1)$

Overdamped response \Rightarrow

Sistema **superamortecido** \Rightarrow $(\zeta > 1)$



```

Matlab:
>> num=9;
>> den=conv([1 0],[1 9 9]);
>>
[r,p,k]=residue(num,den)
r =
    0.1708
   -1.1708
    1.0000
p =
   -7.8541
   -1.1459
    0
k =
    []
>> ftmf=tf(9,[1 9 9]);
>> pzmap(ftmf)
>> step(ftmf)
    
```

Respostas transitorias

Sistemas de 2a-orden:

$$R(s) = \frac{1}{s} \rightarrow \boxed{\frac{FTMF(s)}{s^2 + as + b}} \rightarrow Y(s)$$

3 casos:

1) Raízes reais:

- a) Distintas;
- b) Iguais

2) Raízes complexas:

- a) Distante de $j\omega$;
- b) Sobre $j\omega$.

$$G(s) = \frac{9}{s^2 + 6s + 9}$$

2 pólos reais em $-p_1$ e $-p_2$

$$p_{1,2} = -3$$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$$

$$w_n^2 = 9 \rightarrow w_n = 3$$

$$\zeta = \frac{6}{6} = 1$$

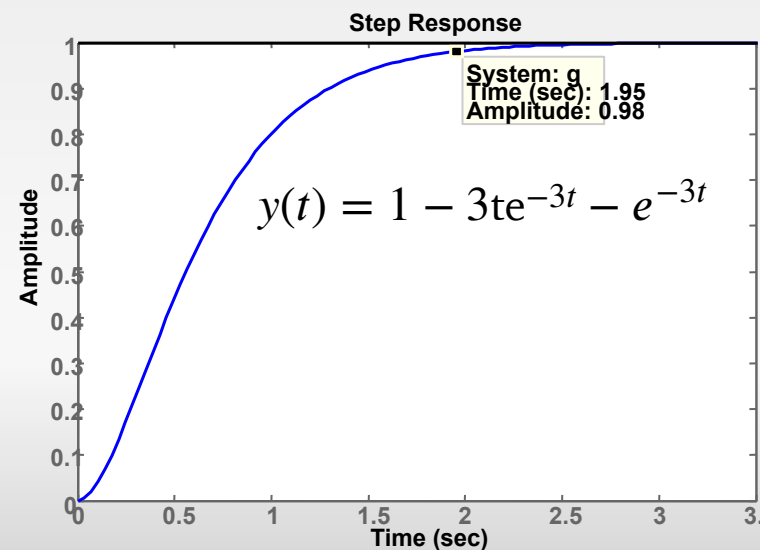
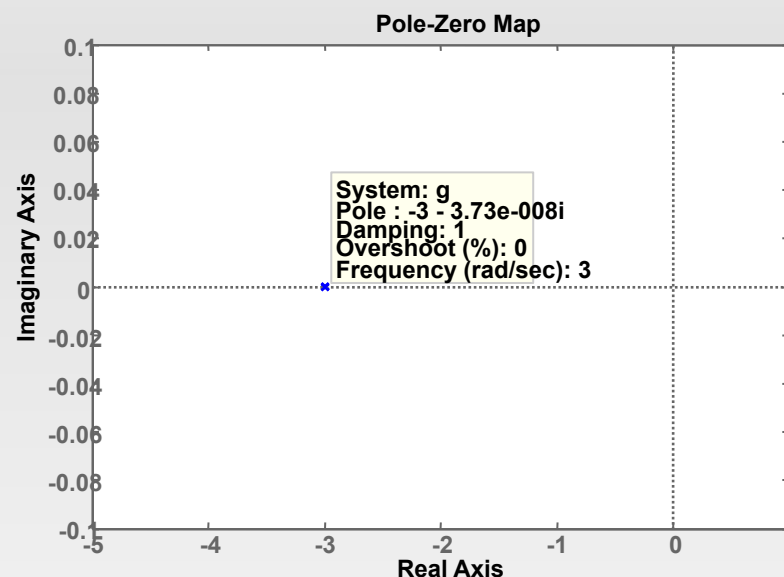
$$y(t) = K_0 + K_1 e^{-at} + K_2 t e^{-at}$$

$$y(t) = 1 - 3te^{-3t} - e^{-3t}$$

Amortecimento crítico \Rightarrow
($\zeta = 1$)

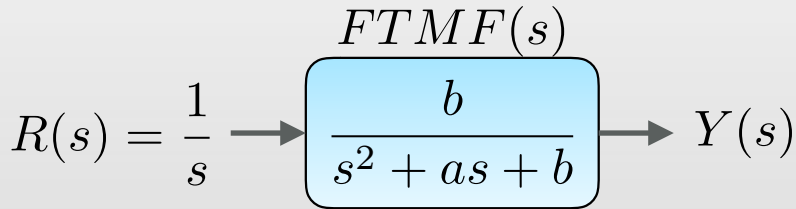
Critically damped \Rightarrow

Amortecimento crítico \Rightarrow ($\zeta = 1$)



Respostas transitorias

Sistemas de 2a-orden:



3 casos:

1) Raízes reais:

a) Distintas;

b) Iguais

$$G(s) = \frac{9}{s^2 + 6s + 9}$$

2 pólos reais em $-p_1$ e $-p_2$

$$p_{1,2} = -3$$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$$

$$w_n^2 = 9 \rightarrow w_n = 3$$

$$\zeta = \frac{6}{6} = 1$$

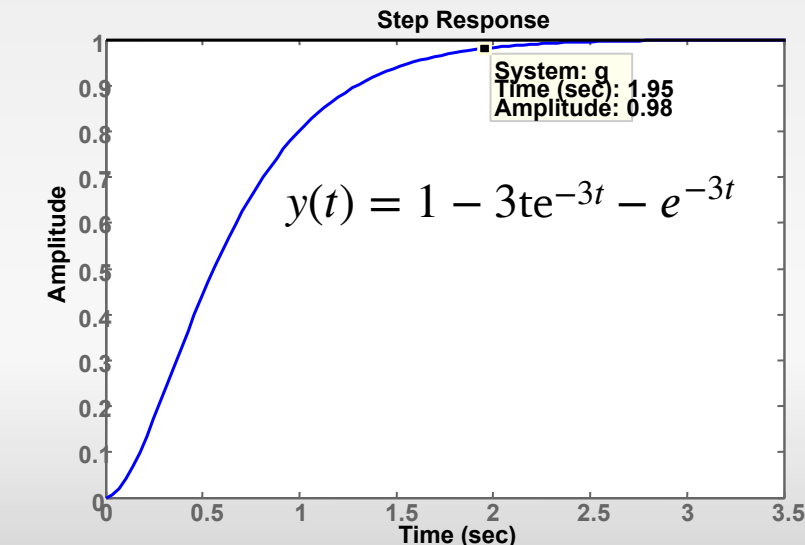
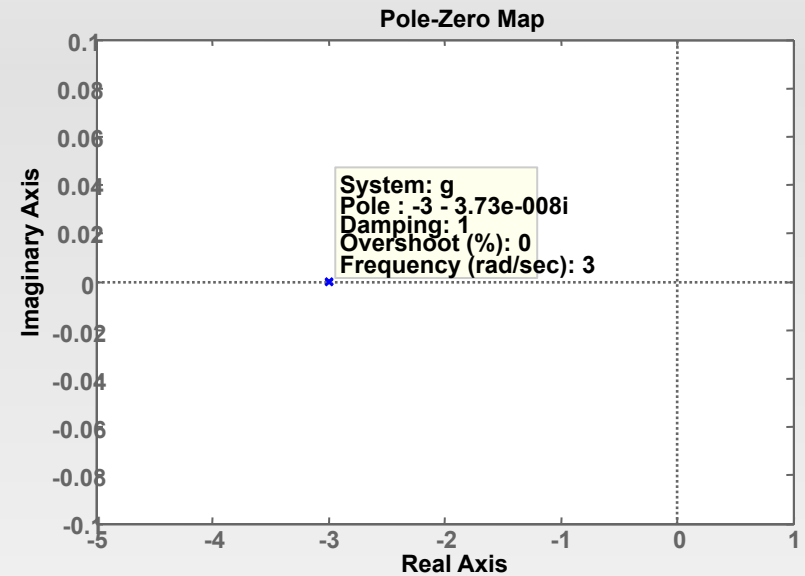
$$y(t) = K_0 + K_1 e^{-at} + K_2 t e^{-at}$$

$$y(t) = 1 - 3te^{-3t} - e^{-3t}$$

Amortecimento crítico \rightarrow
 $(\zeta = 1)$

Critically damped \rightarrow

Amortecimento crítico $\rightarrow (\zeta = 1)$



```
Matlab:
>> num=9;
>> den=conv([1 0],[1 6 9]);
>>
>> [r,p,k]=residue(num,den)
r =
    -1
    -3
     1
p =
    -3
    -3
     0
k =
     []
>> ftmf=tf(9,[1 6 9]);
>> pzmap(ftmf)
>> step(ftmf)
```

Respostas transitorias

Sistemas de 2a-orden:

$$R(s) = \frac{1}{s} \rightarrow \left[\frac{FTMF(s)}{s^2 + as + b} \right] \rightarrow Y(s)$$

3 casos:

- 1) Raízes reais:
 - a) Distintas;
 - b) Iguais
- 2) Raízes complexas:
 - a) Distante de $j\omega$;
 - b) Sobre $j\omega$.

$$G(s) = \frac{9}{s^2 + 2s + 9}$$

2 pólos complexos em $-p_1$ e $-p_2$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$$

$$p_{1,2} = \sigma \pm j w_d \quad w_d = w_n \sin(\alpha) = w_n \sqrt{1 - \zeta^2}$$

$$p_{1,2} = -1 \pm j\sqrt{8}$$

$$w_n^2 = 9 \rightarrow w_n = 3$$

$$\zeta = \frac{2}{6} = 0,3333$$

$$y(t) = 1 - e^{-t} \left[\cos(\sqrt{8}t) + \frac{\sqrt{8}}{8} \sin(\sqrt{8}t) \right]$$

$$y(t) = 1 - 1,06 e^{-t} \cos(\sqrt{8}t - 19,47^\circ)$$

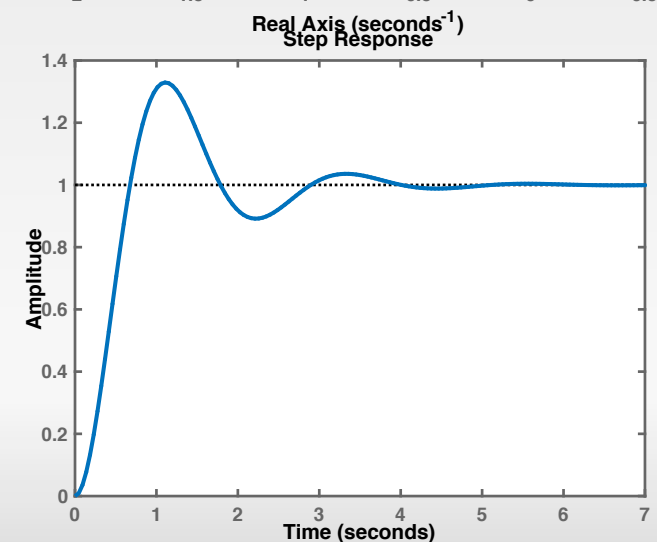
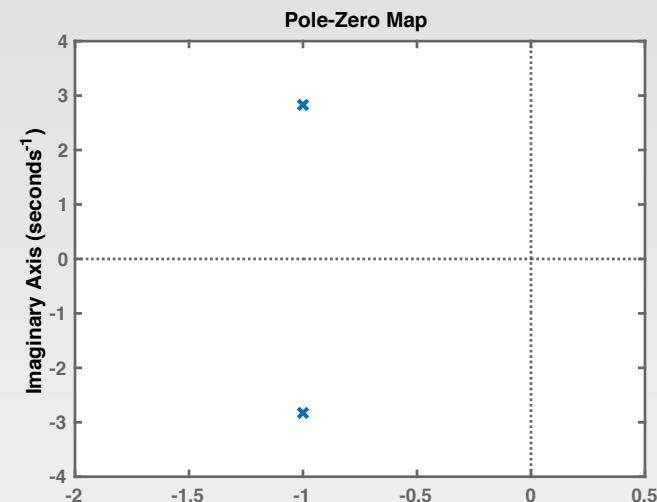
$$y(t) = 1 - A e^{-\sigma\%t} \cos(w_d t - \phi)$$

Sistema subamortecido \Rightarrow
 $(0 < \zeta < 1)$

underdamped response \Rightarrow

Prof. Fernando Passold

Sistema subamortecido $\Rightarrow (0 < \zeta < 1)$



Respostas transitorias

Sistemas de 2a-orden:

$$R(s) = \frac{1}{s} \rightarrow \boxed{\frac{FTMF(s)}{s^2 + as + b}} \rightarrow Y(s)$$

3 casos:

1) Raízes reais:

- a) Distintas;
- b) Iguais

2) Raízes complexas:

- a) Distante de $j\omega$;
- b) Sobre $j\omega$.

$$G(s) = \frac{9}{s^2 + 9}$$

2 pólos complexos em $\pm j\omega_n$:

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$$

$$p_{1,2} = \pm j\sqrt{3}$$

$$w_n^2 = 9 \rightarrow w_n = 3$$

$$\zeta = 0$$

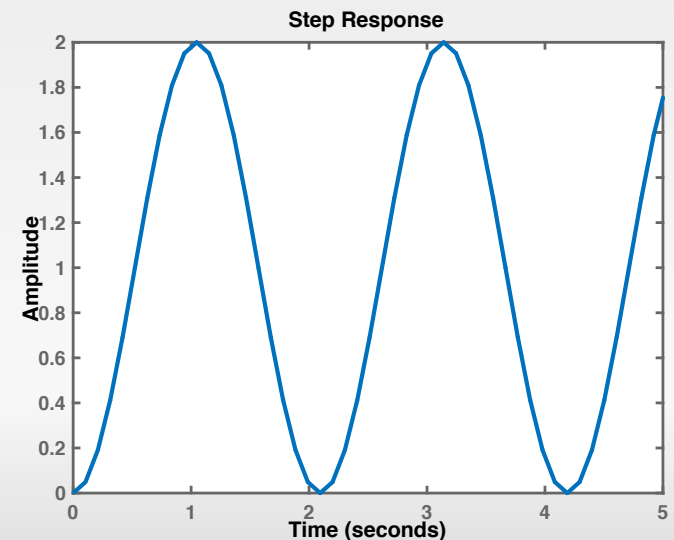
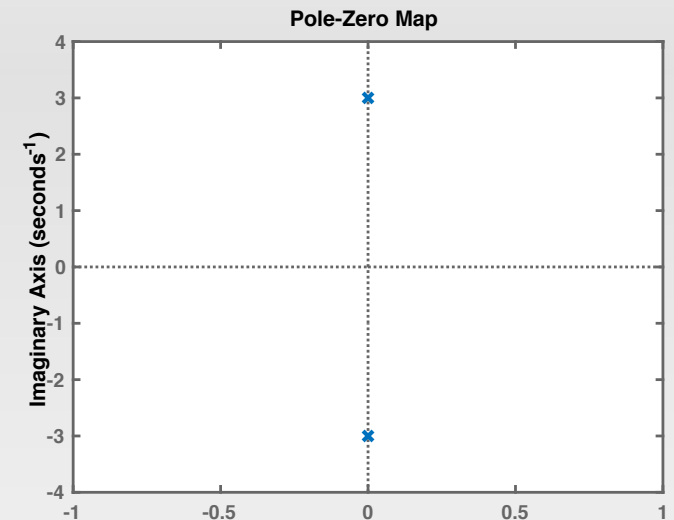
$$y(t) = 1 - \cos(3t)$$

$$y(t) = A \cos(w_1 t - \phi)$$

Sistema oscilatório \Rightarrow
($\zeta = 0$)

undamped response \Rightarrow

Sistema oscilatório \Rightarrow ($\zeta = 0$)



Respostas transitorias

Sistemas de 2a-orden — Resumo:

$$R(s) = \frac{1}{s} \rightarrow \boxed{\frac{b}{s^2 + as + b}} \xrightarrow{FTMF(s)} Y(s)$$

- Super amortecido: $\zeta > 1$

$$\frac{9}{s^2 + 9s + 9} \quad \gg \text{pole(g1)}$$

-7.8541
-1.1459

- Subamortecido: $0 < \zeta < 1$

$$\frac{9}{s^2 + 2s + 9} \quad \gg \text{pole(g2)}$$

-1.0000 + 2.8284i
-1.0000 - 2.8284i

$$y(t) = 1 - \exp(-\zeta\omega_n t) \cdot \left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t) \right]$$

$$\text{pólos em: } s = \sigma \pm j\omega_d \quad \begin{cases} \sigma = \omega_n \zeta; \\ \omega_d = \omega_n \sqrt{1-\zeta^2}; \end{cases}$$

- Oscilatório: $\zeta = 0$

$$\frac{9}{s^2 + 9} \quad \gg \text{pole(g3)}$$

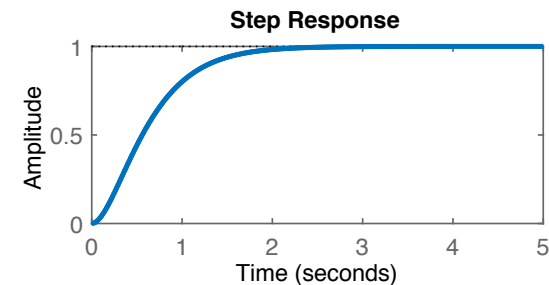
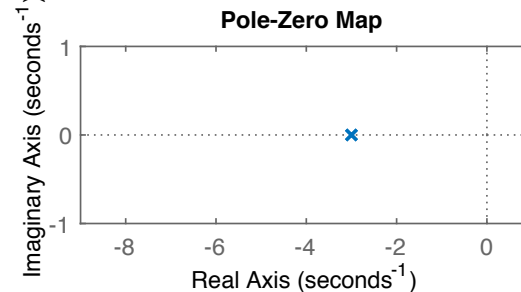
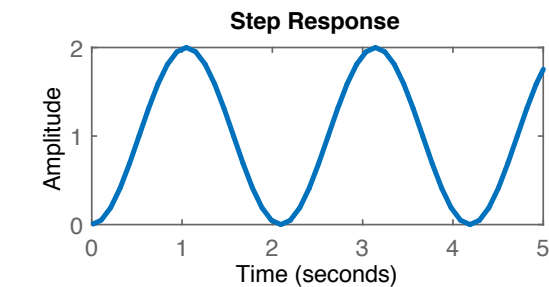
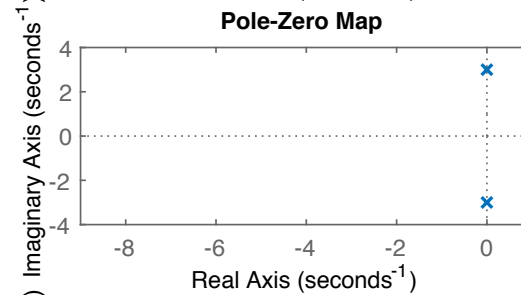
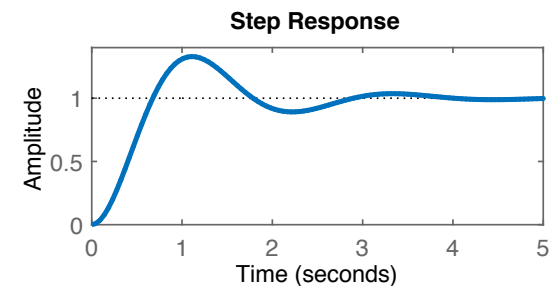
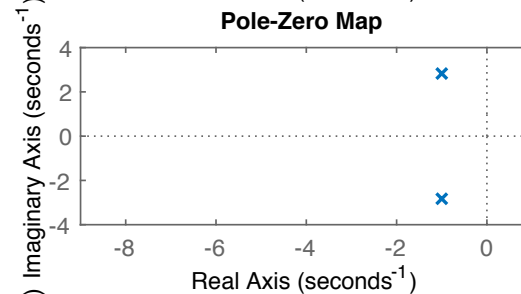
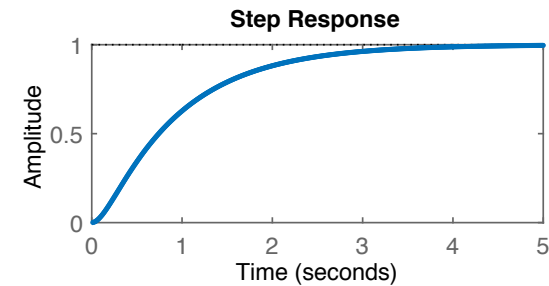
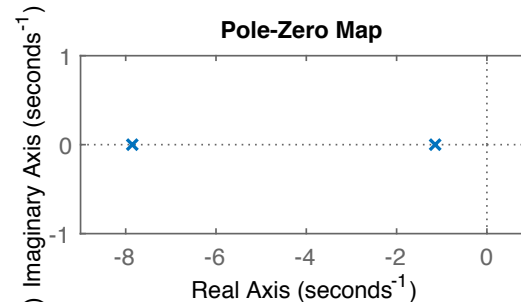
0.0000 + 3.0000i
0.0000 - 3.0000i

- Criticamente amortecido: $\zeta = 1$

$$\frac{9}{s^2 + 6s + 9} \quad \gg \text{pole(g4)}$$

-3.0000 + 0.0000i
-3.0000 - 0.0000i

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{b}{s^2 + as + b}$$



Respostas transitorias

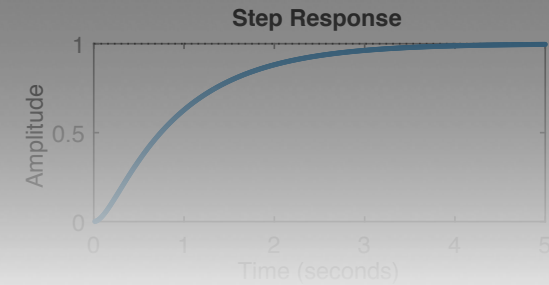
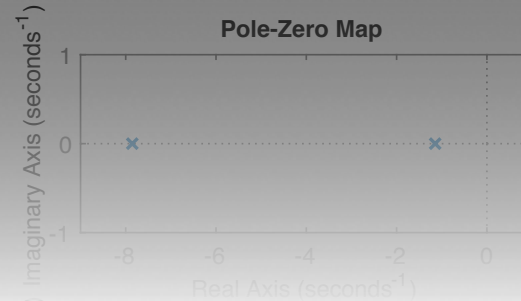
Sistemas de 2a-orden — Resumo:

$$R(s) = \frac{1}{s} \rightarrow \frac{b}{s^2 + as + b} \xrightarrow{FTMF(s)} Y(s)$$

- Super amortecido: $\zeta > 1$

$$\frac{9}{s^2 + 9s + 9} \quad \gg \text{pole}(g1)$$

-7.8541
-1.1459



- Subamortecido: $0 < \zeta < 1$

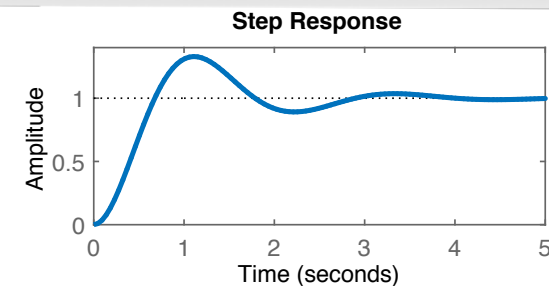
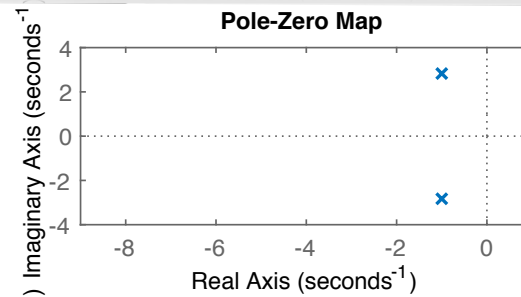
$$\frac{9}{s^2 + 2s + 9} \quad \gg \text{pole}(g2)$$

-1.0000 + 2.8284i
-1.0000 - 2.8284i

$$y(t) = 1 - \exp(-\zeta\omega_n t) \cdot \left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t) \right]$$

pólos em: $s = \sigma \pm j\omega_d$

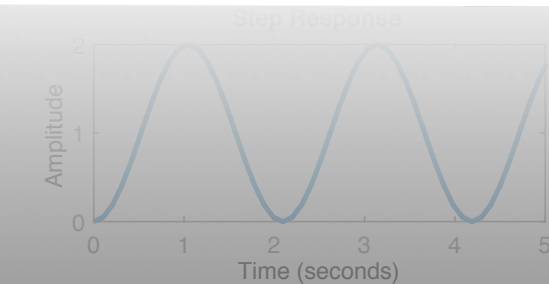
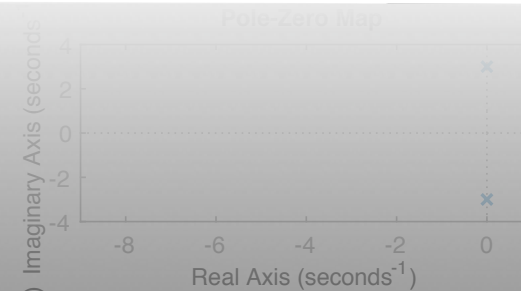
$$\begin{cases} \sigma = \omega_n \zeta; \\ \omega_d = \omega_n \sqrt{1-\zeta^2}; \end{cases}$$



- Oscilatório: $\zeta = 0$

$$\frac{9}{s^2 + 9} \quad \gg \text{pole}(g3)$$

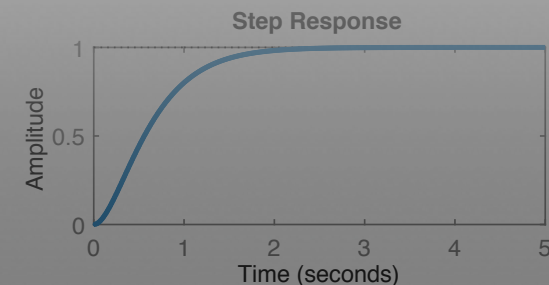
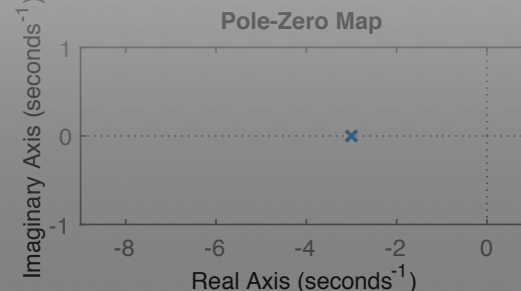
0.0000 + 3.0000i
0.0000 - 3.0000i



- Criticamente amortecido: $\zeta = 1$

$$\frac{9}{s^2 + 6s + 9} \quad \gg \text{pole}(g4)$$

-3.0000 + 0.0000i
-3.0000 - 0.0000i



$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{b}{s^2 + as + b}$$

Sistemas de 2ª ordem — Detalhes

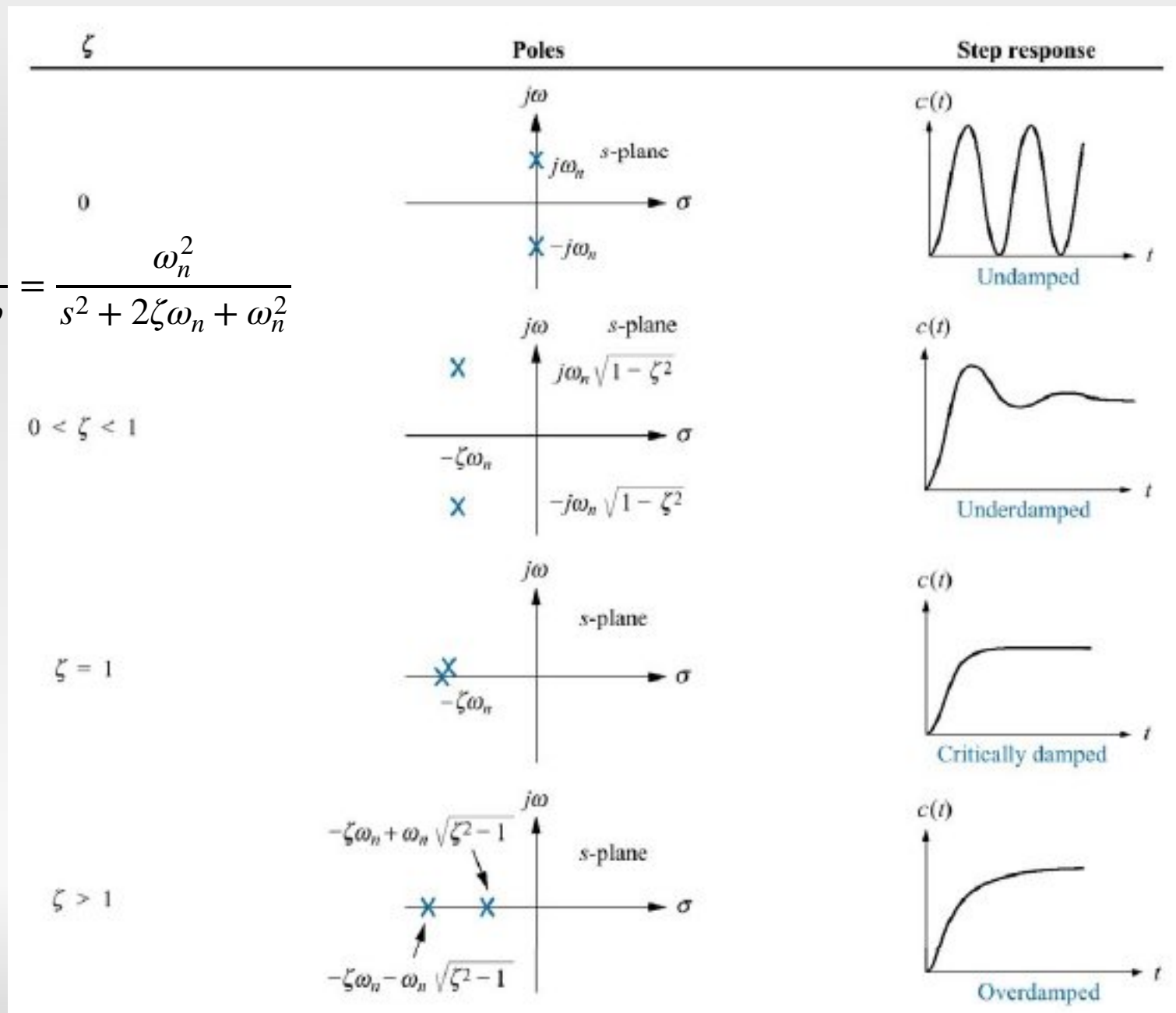
$$FTMF(s) = \frac{b}{s^2 + as + b} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- **Frequência natural de oscilação, ω_n** : frequência de oscilação do sistema sem amortecimento \Rightarrow polos puramente imaginários: $a = 0$, polos sobre $j\omega$ em: $\pm j\sqrt{b}$; $\omega_n = \sqrt{b}$; $b = \omega_n^2$.
- **Coeficiente de amortecimento, ζ** : os polos complexos contêm parte real, σ igual a $-a/2$. A magnitude deste coeficiente modula o decaimento exponencial:

$$\zeta = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n}; \quad a = 2\zeta\omega_n$$

Sistemas de 2ª ordem — Detalhes

$$FTMF(s) = \frac{b}{s^2 + as + b} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



Sistema subamortecido

- Resposta ao degrau:

$$C(s) = R(s) Y(s) = \frac{K}{s(s+p_1)(s+p_2)} = k \frac{w_n^2}{s(s^2 + 2\zeta w_n s + w_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta w_n s + w_n^2}$$

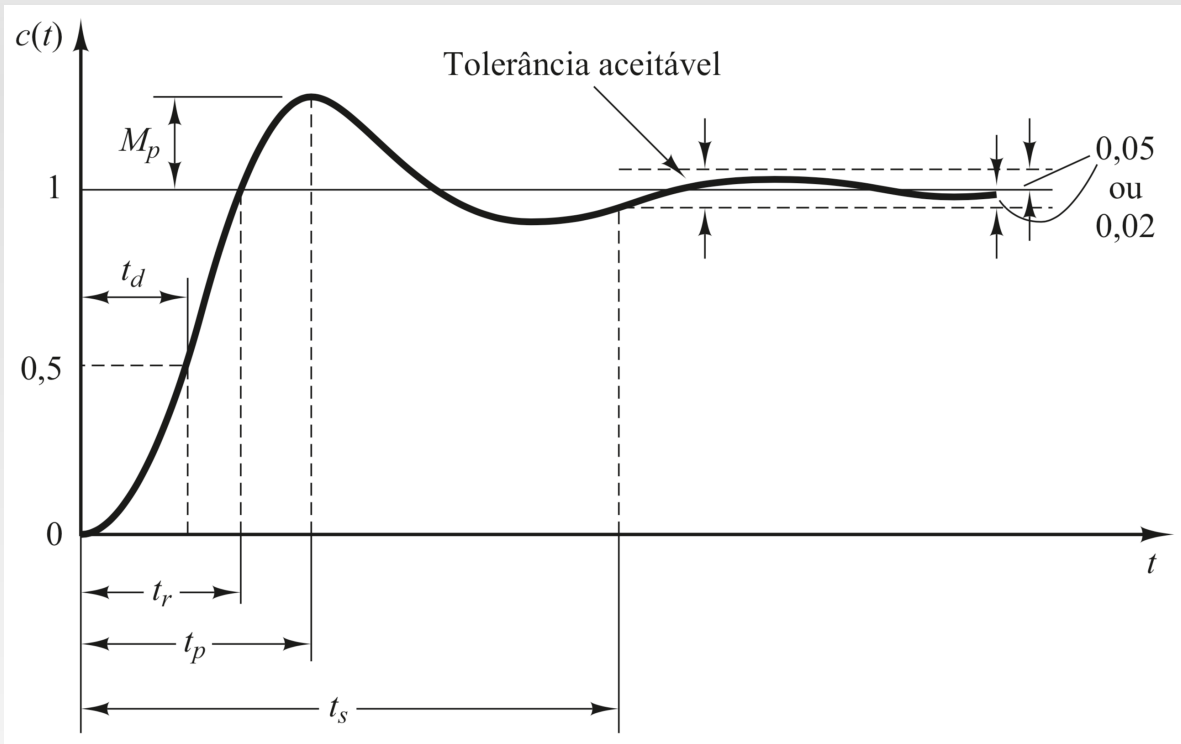
- Assumindo: $0 < \zeta < 1$ (sistema subamortecido):

$$C(s) = \frac{1}{s} - \frac{(s + \zeta w_n) + \frac{\zeta}{\sqrt{1-\zeta^2}} w_n \sqrt{1-\zeta^2}}{(s + \zeta w_n)^2 + w_n^2 (1-\zeta^2)}$$

$$c(t) = 1 - e^{-\zeta w_n t} \left(\cos w_n \sqrt{1-\zeta^2} t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin w_n \sqrt{1-\zeta^2} t \right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} \cos(w_n \sqrt{1-\zeta^2} t - \phi) \quad \text{where } \phi = \tan^{-1} (\zeta / \sqrt{1-\zeta^2})$$

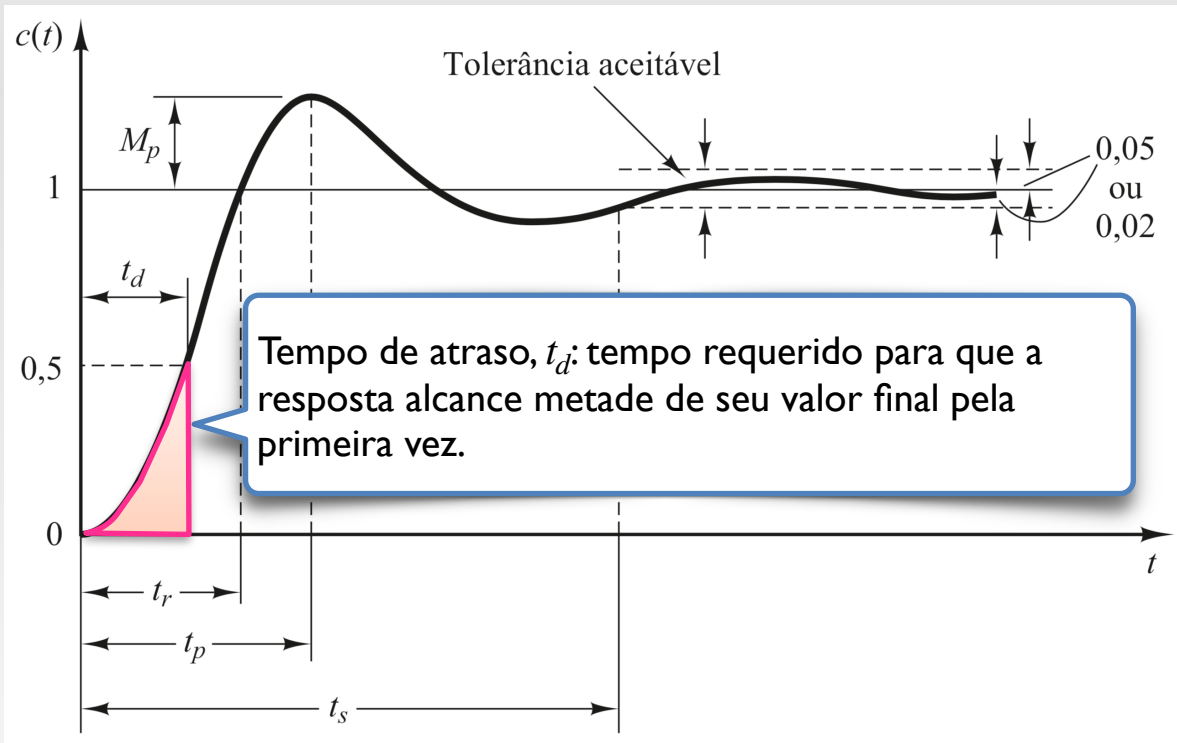
Respostas de Sistemas de 2ª ordem...



Parâmetros temporais:

1. Tempo de subida, t_r ;
2. Tempo de pico, t_p ;
3. Máximo sobressinal, M_p ,
% OS (overshoot);
4. Tempo de acomodação, t_s ,
(settling time);
5. Tempo de atraso, t_d .

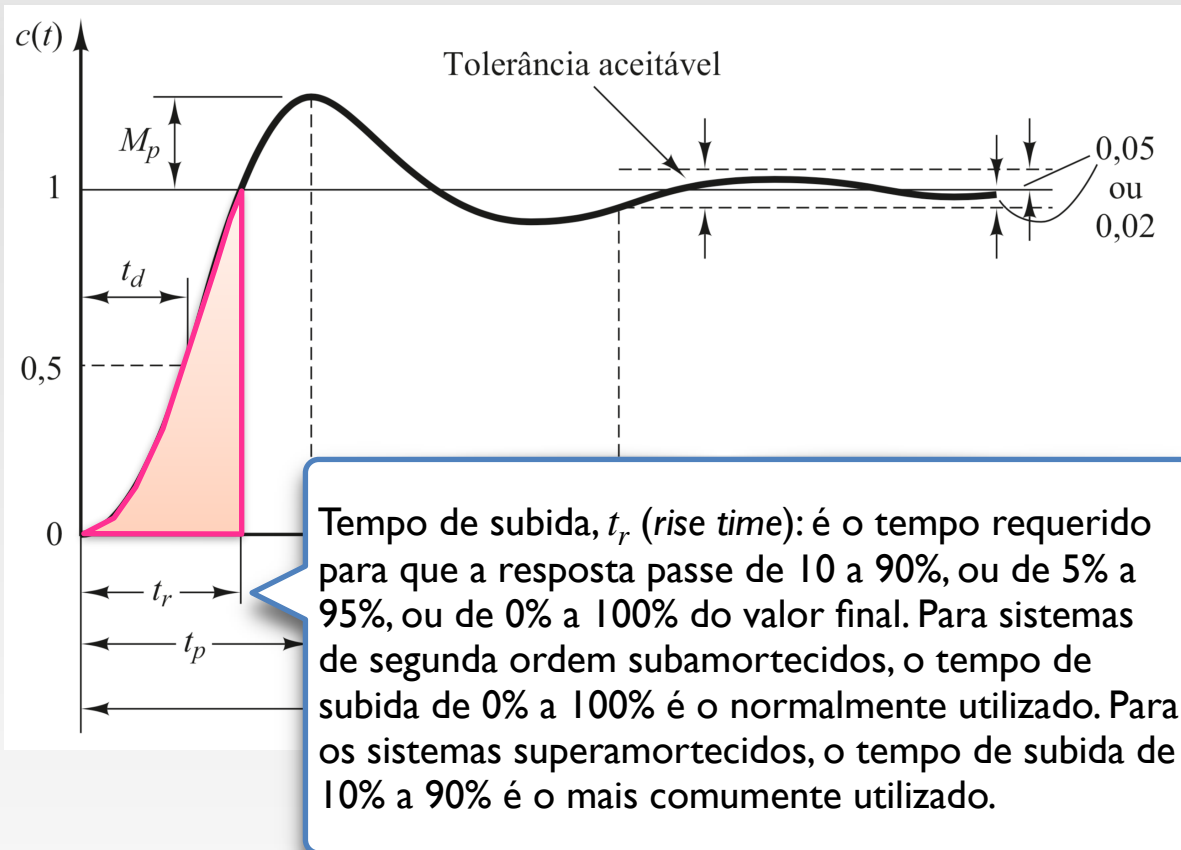
Respostas de Sistemas de 2ª ordem...



Parâmetros temporais:

1. Tempo de subida, t_r ;
2. Tempo de pico, t_p ;
3. Máximo sobressinal, M_p ,
 $\% OS$ (overshoot);
4. Tempo de acomodação, t_s ,
(settling time);
5. Tempo de atraso, t_d .

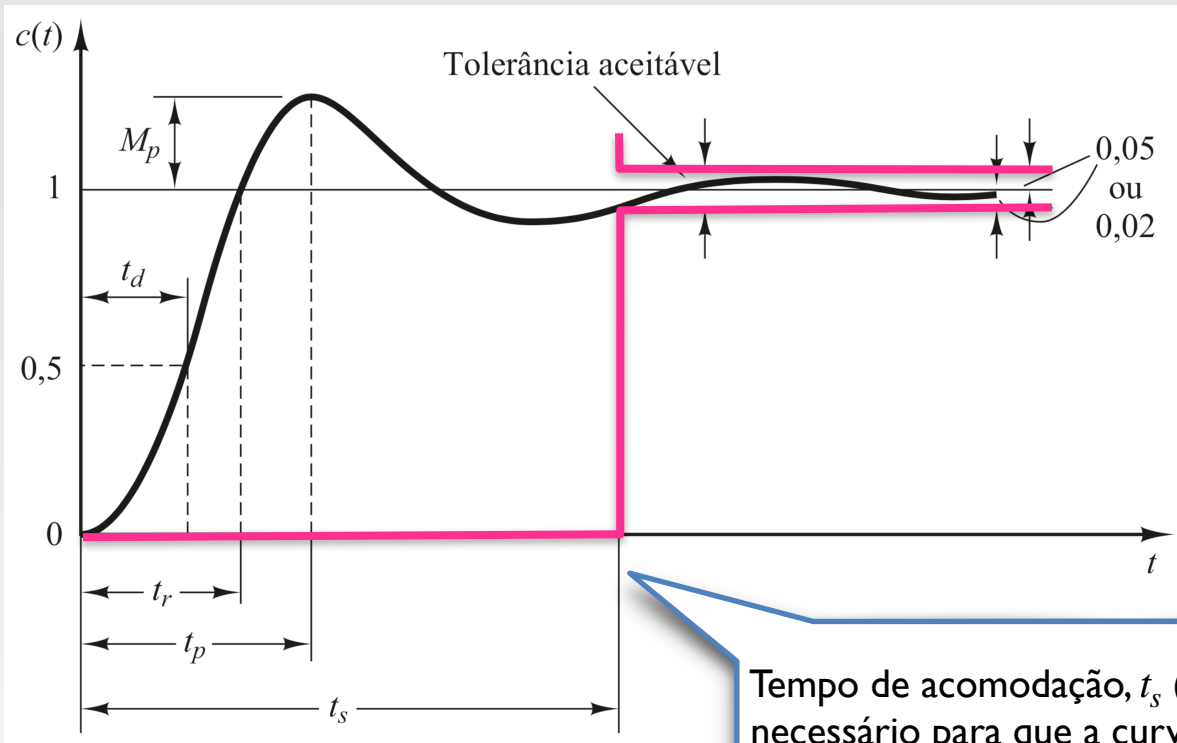
Respostas de Sistemas de 2ª ordem...



Parâmetros temporais:

1. Tempo de subida, t_r ;
2. Tempo de pico, t_p ;
3. Máximo sobressinal, M_p ,
% *OS* (overshoot);
4. Tempo de acomodação, t_s ,
(settling time);
5. Tempo de atraso, t_d .

Respostas de Sistemas de 2ª ordem...

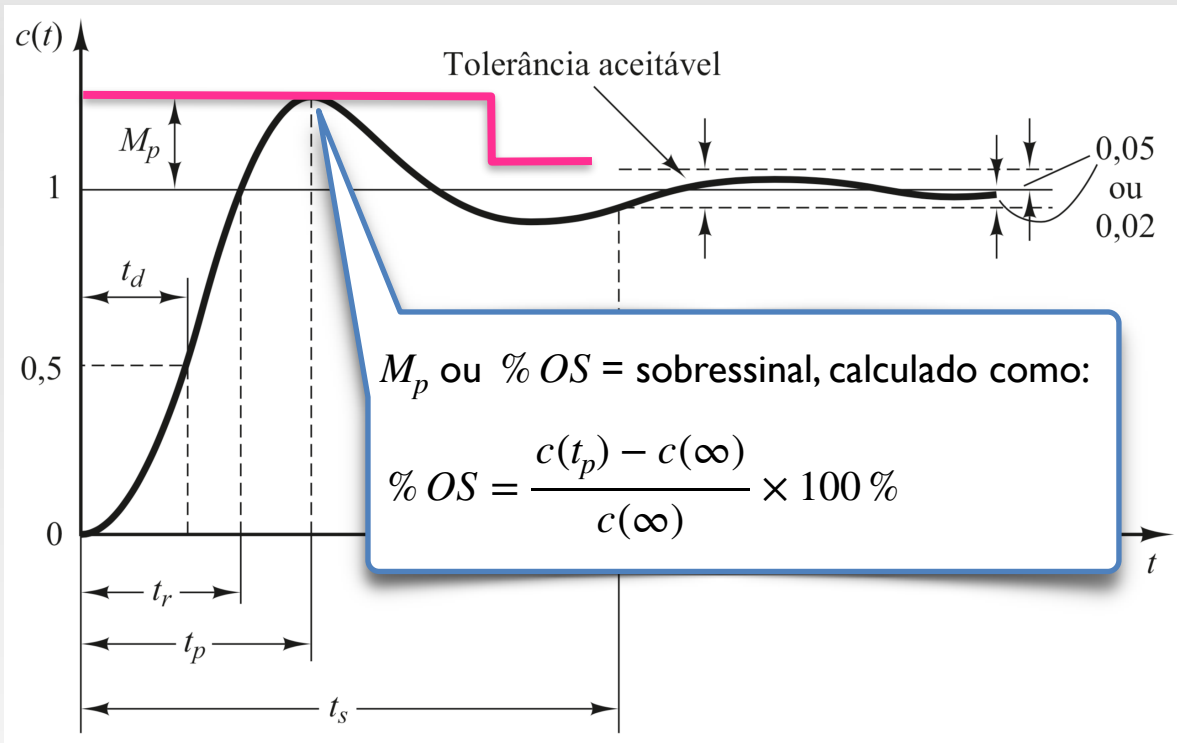


Parâmetros temporais:

1. Tempo de subida, t_r ;
2. Tempo de pico, t_p ;
3. Máximo sobressinal, M_p ,
% OS (overshoot);
4. Tempo de acomodação, t_s ,
(settling time);
5. Tempo de atraso, t_d .

Tempo de acomodação, t_s (*settling time*): é o tempo necessário para que a curva de resposta alcance valores em uma faixa (geralmente de 2% ou 5%) em torno do valor final, aí permanecendo indefinidamente. O tempo de acomodação está relacionado à maior constante de tempo do sistema de controle.

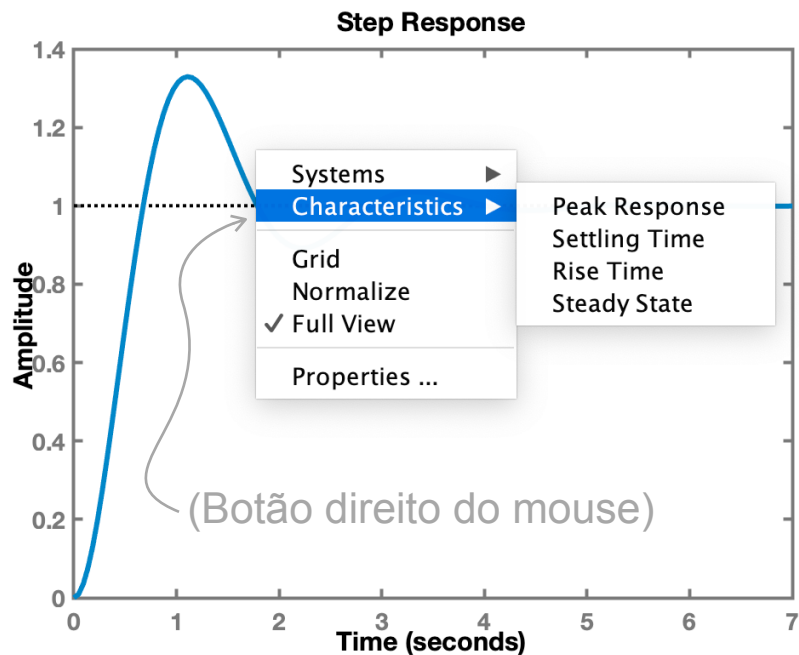
Respostas de Sistemas de 2ª ordem...



Parâmetros temporais:

1. Tempo de subida, t_r ;
2. Tempo de pico, t_p ;
3. Máximo sobressinal, M_p ,
 $\% OS$ (overshoot);
4. Tempo de acomodação, t_s ,
(settling time);
5. Tempo de atraso, t_d .

Respostas de Sistemas de 2ª ordem...



Parâmetros temporais:

1. Tempo de subida, t_r ;
2. Tempo de pico, t_p ;
3. Máximo sobressinal, M_p ,
% *OS* (*overshoot*);
4. Tempo de acomodação, t_s ,
(*settling time*);
5. Tempo de atraso, t_d .

Matlab:

```
>> G=tf(9, [1 2 9]);  
>> zpk(G)
```

$$\frac{9}{s^2 + 2s + 9}$$

Continuous-time zero/pole/gain model.

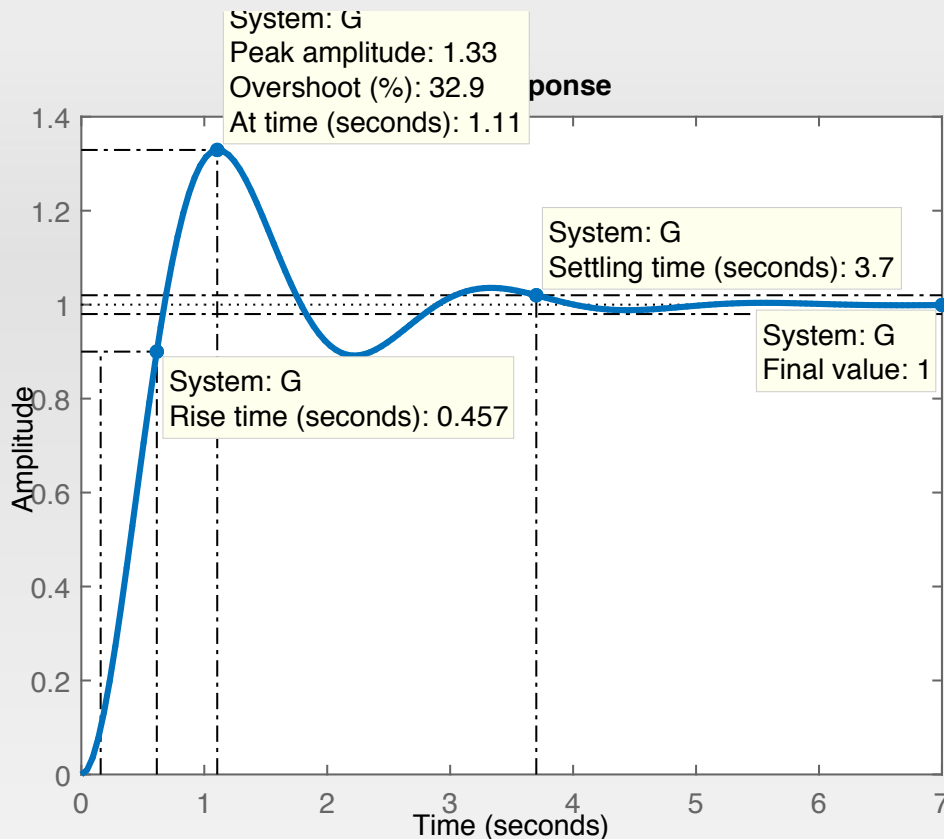
```
>> step(G)
```

```
>> stepinfo(G)
```

```
ans =  
  struct with fields:  
  
    RiseTime: 0.4568  
  SettlingTime: 3.7005  
  SettlingMin: 0.8916  
  SettlingMax: 1.3293  
    Overshoot: 32.9277  
  Undershoot: 0  
      Peak: 1.3293  
    PeakTime: 1.1052
```

```
>>
```

Respostas de Sistemas de 2ª ordem...



Matlab:

```
>> G=tf(9, [1 2 9]);
```

```
>> zpk(G)
```

$$\frac{9}{s^2 + 2s + 9}$$

Continuous-time zero/pole/gain model.

```
>> step(C)
```

Parâmetros temporais:

1. Tempo de subida, t_r ;
2. Tempo de pico, t_p ;
3. Máximo sobressinal, M_p ,
 $\% OS$ (overshoot);
4. Tempo de acomodação, t_s ,
(settling time);
5. Tempo de atraso, t_d .

```
>> stepinfo(C)
```

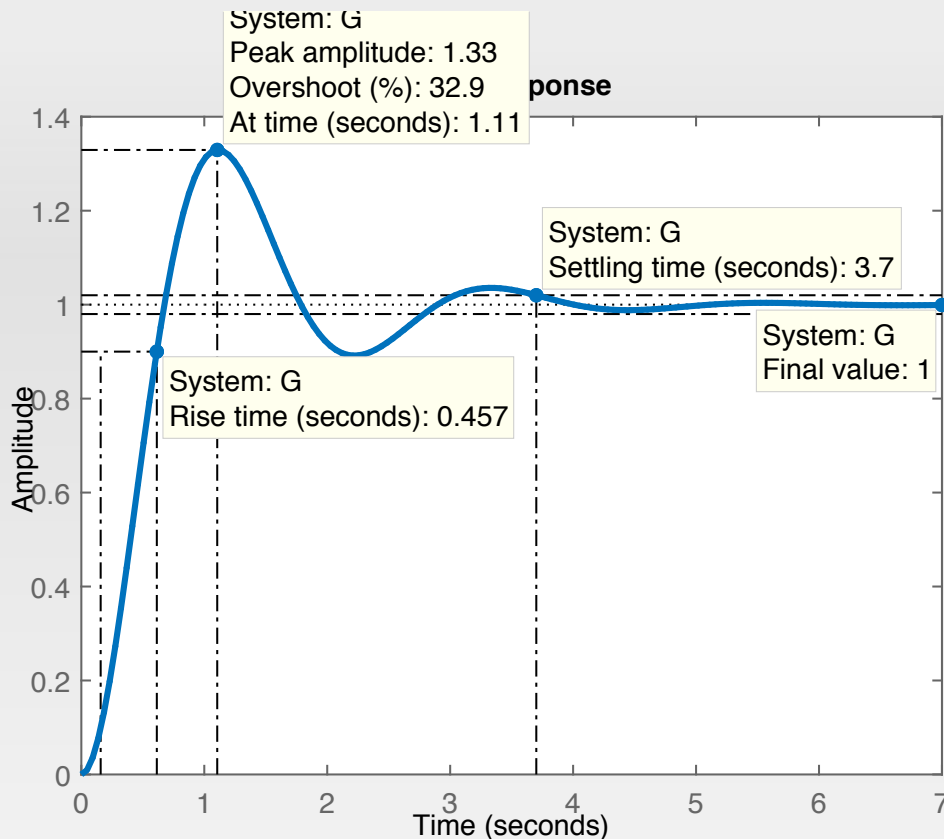
```
ans =
```

```
struct with fields:
```

```
    RiseTime: 0.4568  
    SettlingTime: 3.7005  
    SettlingMin: 0.8916  
    SettlingMax: 1.3293  
    Overshoot: 32.9277  
    Undershoot: 0  
    Peak: 1.3293  
    PeakTime: 1.1052
```

```
>>
```

Respostas de Sistemas de 2ª ordem subamortecidos ($0 < \zeta < 1$)



Tempo de pico, t_p (*peak time*): se obtêm

resolvendo $\frac{\partial y}{\partial t} = 0$:

$$0 = \frac{dy}{dt} = L^{-1}\{\dot{c}(t)\} = L^{-1}\left\{s \cdot \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\}$$

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t)$$

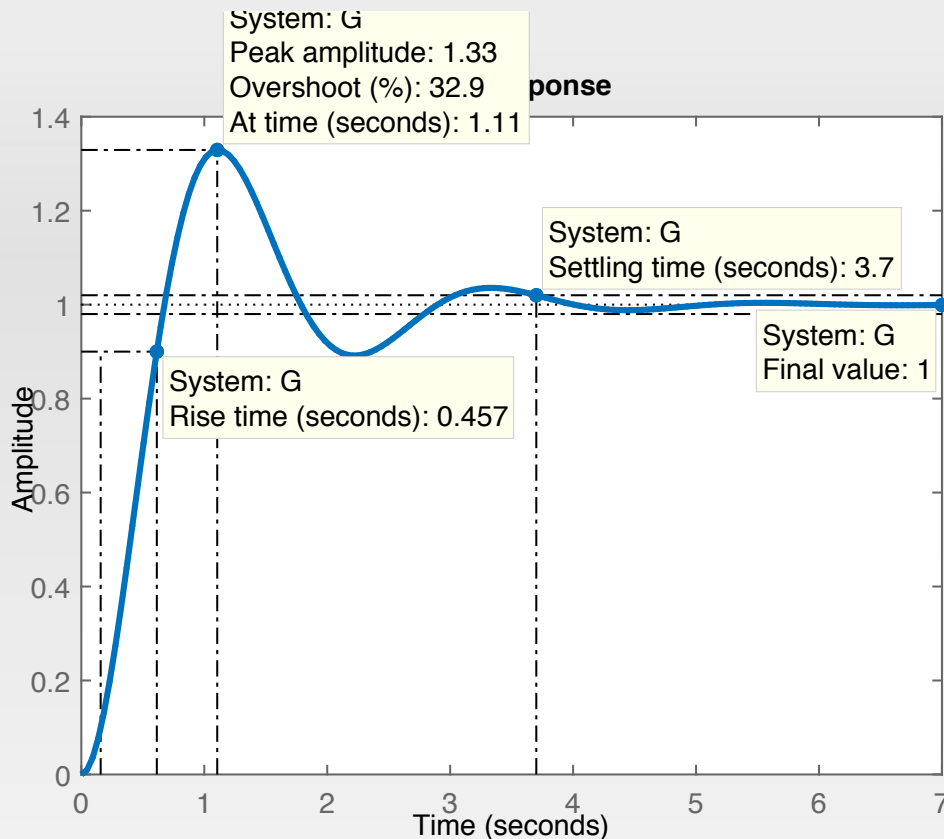
$$\omega_n \sqrt{1-\zeta^2} t = n\pi$$

$n = 0$: primeiro ponto da curva de resposta (inclinação zero);

$n = 1$: primeiro pico, equivale a $t = t_p$:

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Respostas de Sistemas de 2ª ordem subamortecidos ($0 < \zeta < 1$)



Percentual de overshoot (sobressinal), % OS:

$$\%OS = \frac{c_{max} - c_{final}}{c_{final}} \times 100\%$$

$c_{max} = c(t_p)$; uso da equação anterior sobre:

$$c(t) = 1 - e^{-\zeta \omega_n t} \left(\cos(\omega_n \sqrt{1-\zeta^2} t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t) \right)$$

$$c_{max} = c(t_p) = 1 - e^{-(\zeta \pi / \sqrt{1-\zeta^2})} \left(\cos(\pi) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\pi) \right)$$

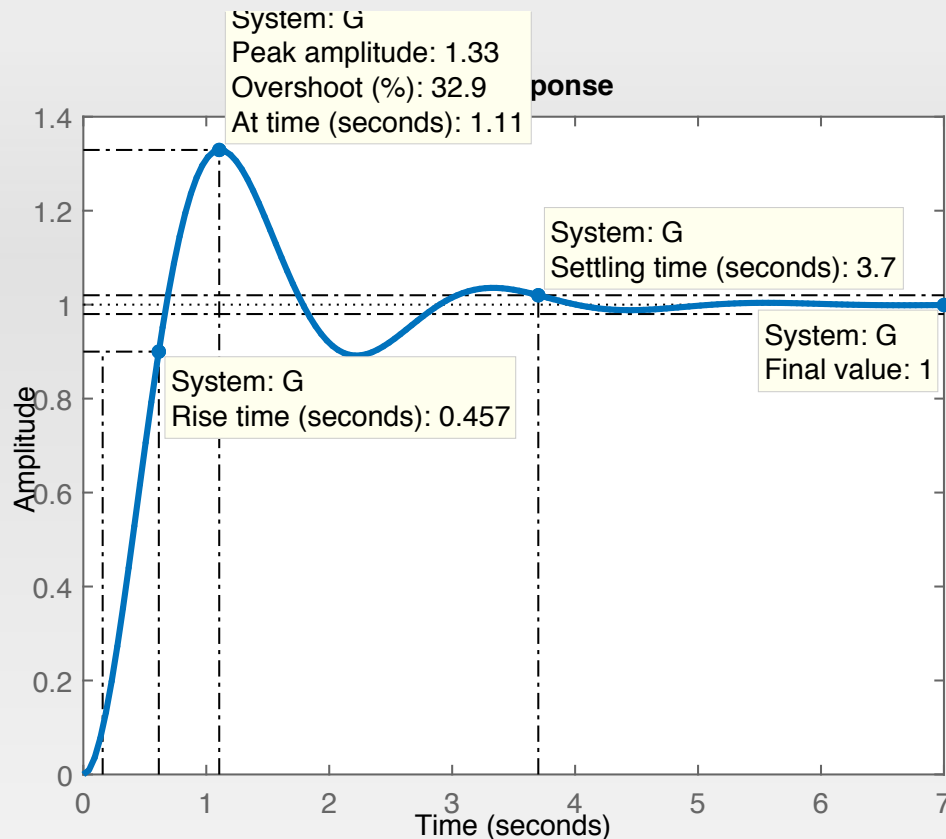
$$c_{max} = 1 + e^{-(\zeta \pi / \sqrt{1-\zeta^2})}$$

$$c_{final} = 1$$

$$\% OS = \exp\left(\frac{-\zeta \pi}{\sqrt{1-\zeta^2}}\right) \times 100\%$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Respostas de Sistemas de 2ª ordem subamortecidos ($0 < \zeta < 1$)



Tempo de assentamento (settling time), t_s ($\pm 2\%$ de tolerância):

A partir de:

$$c(t) = 1 - e^{-\zeta \omega_n t} \left(\cos(\omega_n \sqrt{1-\zeta^2} t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t) \right)$$

Isolamos:

$$e^{-\zeta \omega_n t} \frac{1}{\sqrt{1-\zeta^2}} = 0,02 \quad (4.40)$$

Sabe-se ainda que em $t = t_p$:

$$\cos(\omega_n \sqrt{1-\zeta^2} t - \phi) = 1$$

Resolvendo (4.40) para $t = t_p$:

$$t_s = \frac{-\ln\left(0,02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

O numerador desta eq. varia entre 3,91 à 4,74 enquanto $0 < \zeta < 1$, então uma aproximação seria:

$$t_s = \frac{4}{\zeta\omega_n}$$

Respostas de Sistemas de 2ª ordem subamortecidos ($0 < \zeta < 1$) — Resumo

Equações do sistema em MF:

$$FTMF(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{b}{s^2 + as + b}$$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s + p_1)(s + p_2)} = K \left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right)$$

pólos em: $s = \sigma \pm j\omega_d$ ou: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$

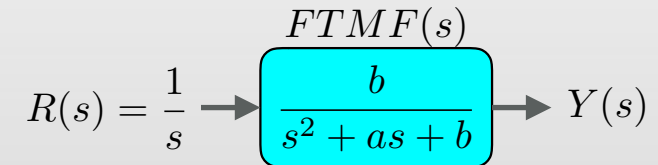
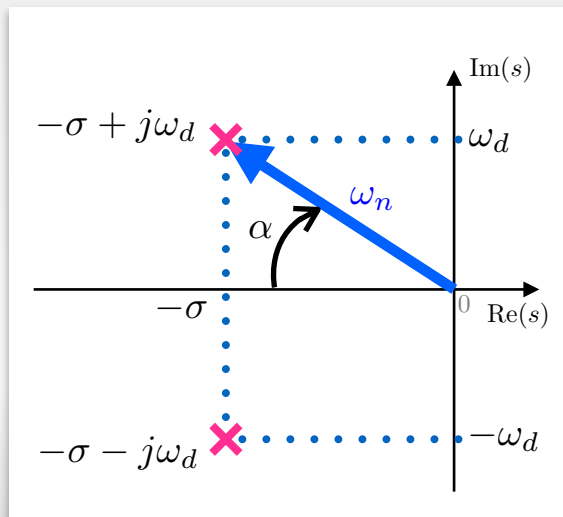
$$\sigma = \omega_n \cos(\alpha) = \omega_n \zeta;$$

$$\omega_d = \omega_n \sin(\alpha) = \omega_n \sqrt{1 - \zeta^2};$$

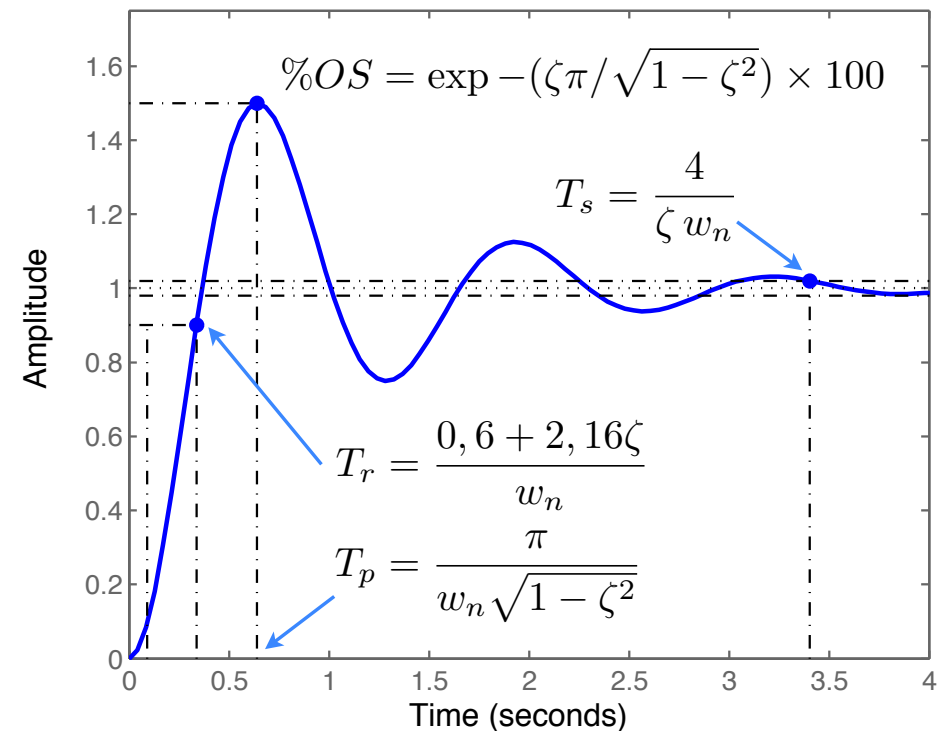
$$\zeta = \cos(\alpha);$$

$$\sin(\alpha) = \sqrt{1 - \zeta^2};$$

$$0 < \zeta < 1$$



Step Response



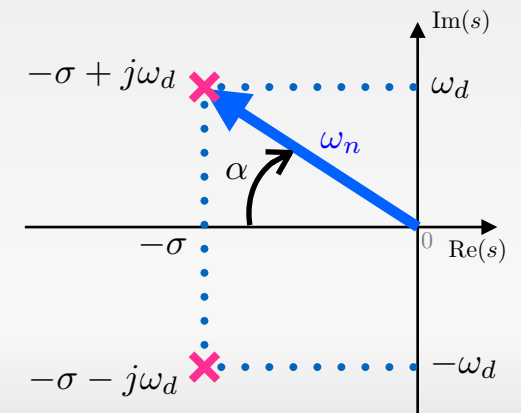
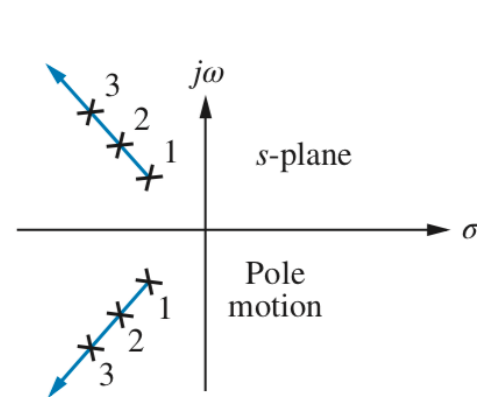
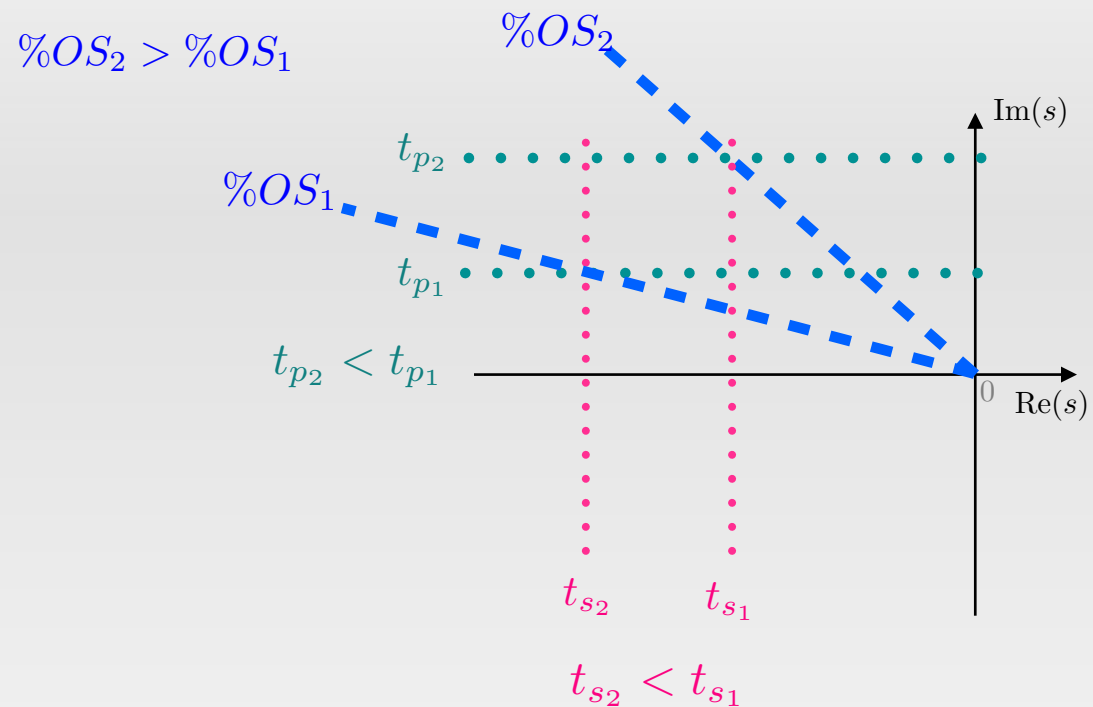
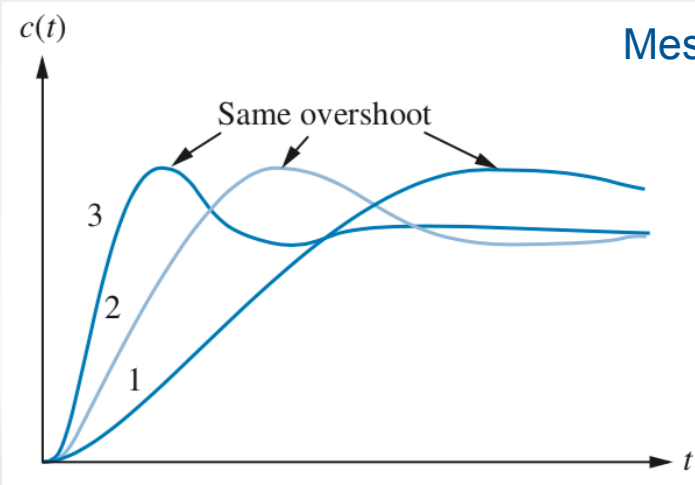
Linhas guia no plano-s

$$\%OS = \exp\left(-\zeta\pi/\sqrt{1-\zeta^2}\right) \times 100$$

$$\zeta = \cos(\theta)$$



Mesmas linhas radiais:
= mesmo ζ

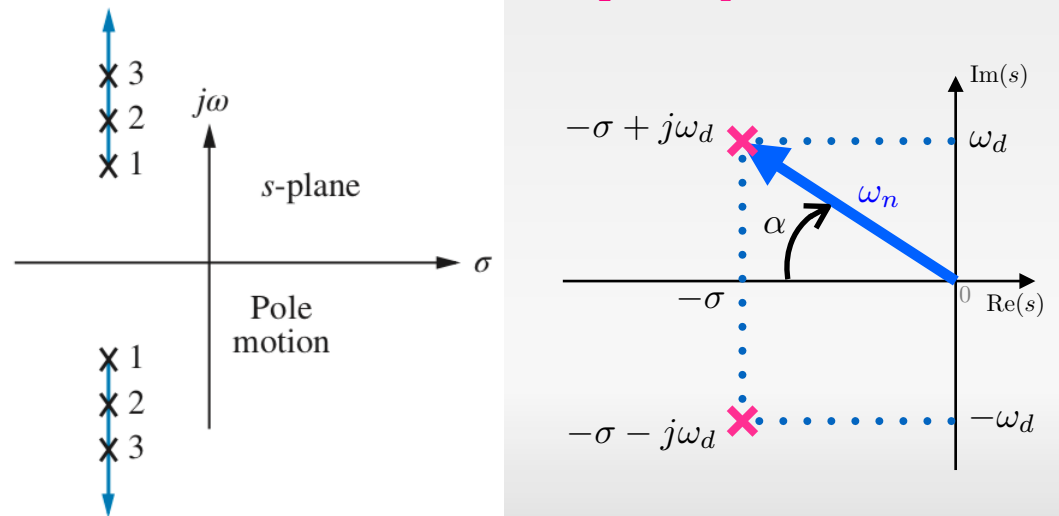
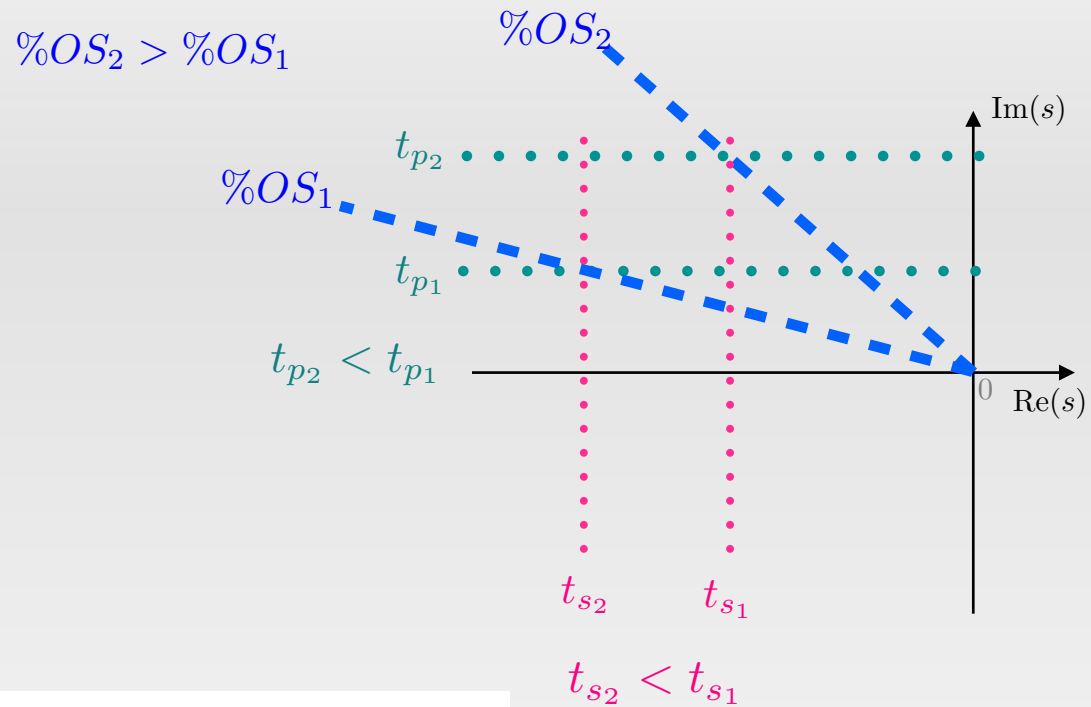
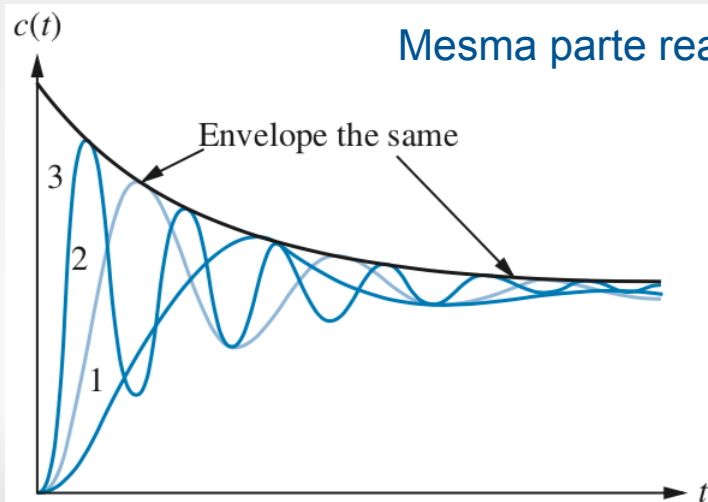


Linhas guia no plano-s

$$t_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\pi}{\omega_d}$$

t_p
 \Downarrow

Inversamente proporcional à parte imaginária do pólo.

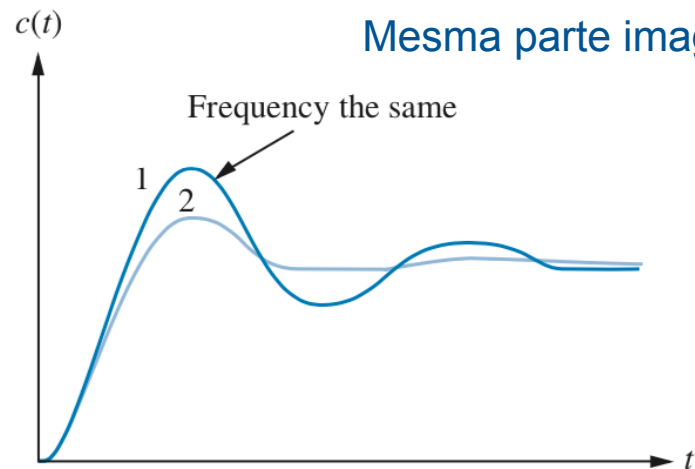


Linhas guia no plano-s

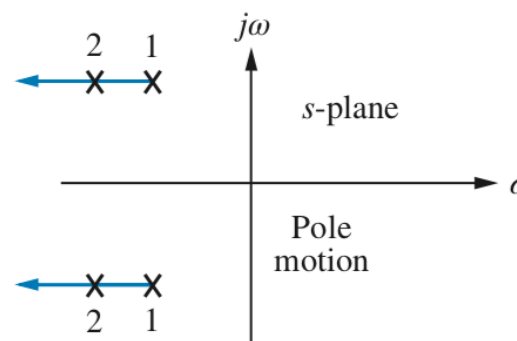
$$t_s = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma}$$

t_s
↓

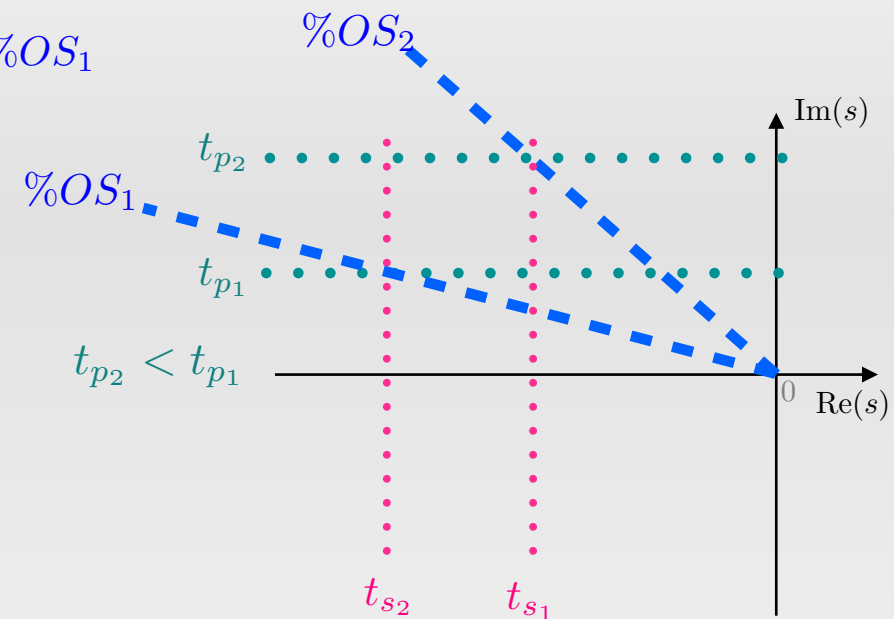
Inversamente
proporcional à parte real
do pólo



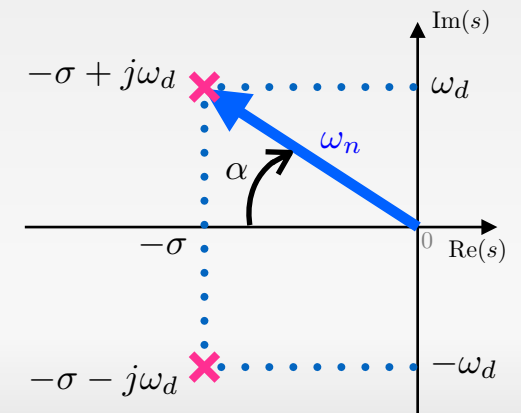
Mesma parte imaginária ($= \omega_d$)



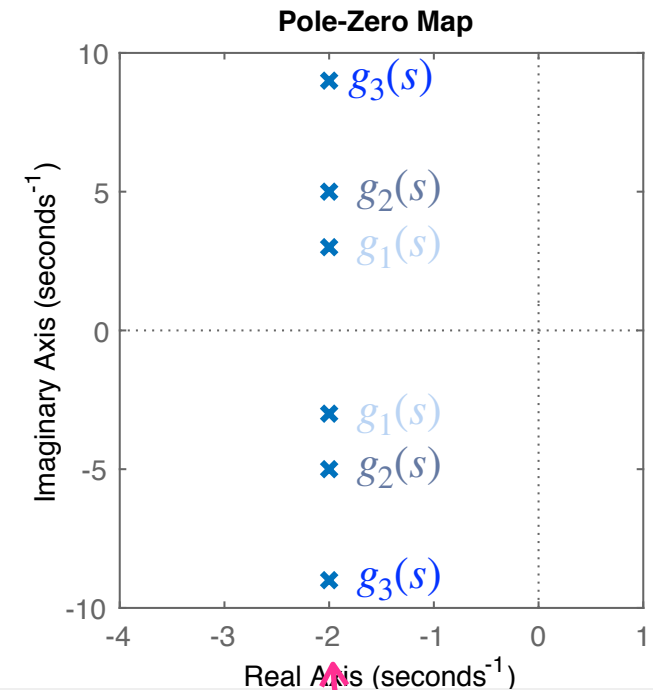
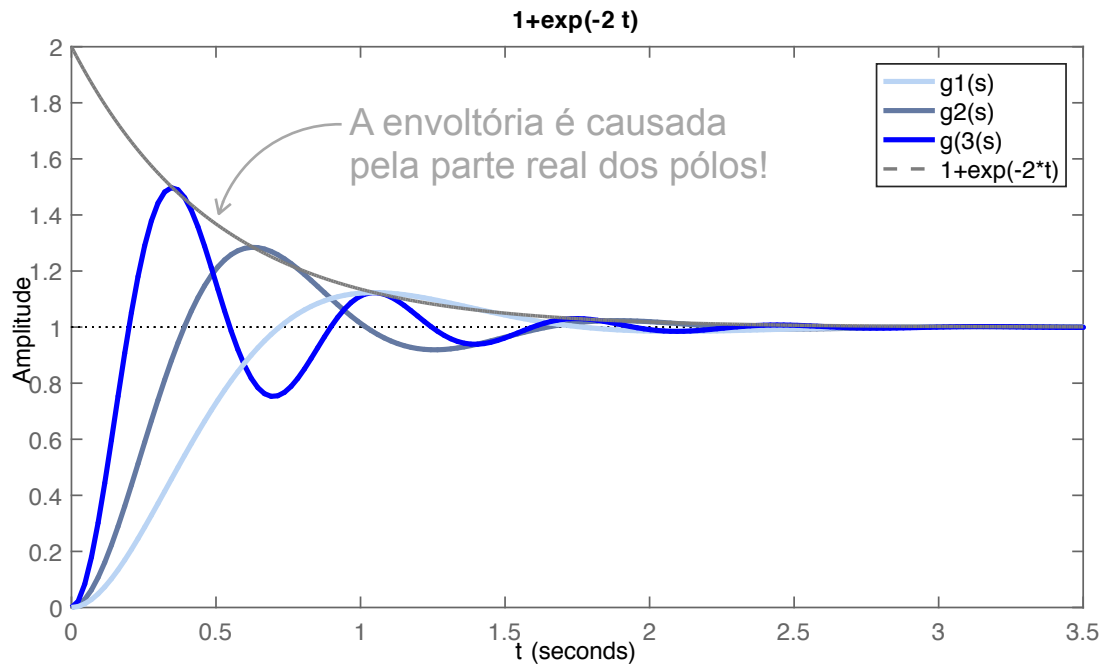
$$\%OS_2 > \%OS_1$$



$$t_{s2} < t_{s1}$$



Sistemas subamortecidos (mesmo σ)



$$g_1(s) = \frac{13}{(s^2+4s+13)} = \frac{13}{(s+2+j3)(s+2-j3)}$$

$$= \frac{(3,6056)^2}{s^2+2(0,5547)(3,6056)s+(3,6056)^2}$$

$$g_2(s) = \frac{9}{(s^2+2s+9)} = \frac{13}{(s+2+j5)(s+2-j5)}$$

$$= \frac{(5,3852)^2}{s^2+2(0,3714)(5,3852)s+(5,3852)^2}$$

$$g_3(s) = \frac{85}{(s^2+4s+85)} = \frac{85}{(s+2+j9)(s+2-j9)}$$

$$= \frac{(9,2195)^2}{s^2+2(0,2169)(9,2195)s+(9,2195)^2}$$

Mesma parte real

Resumo sistemas subamortecidos ($0 < \zeta < 1$)

$$FTMF(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{b}{s^2 + as + b}$$

$$y(t) = 1 - \exp(-\zeta\omega_n t) \cdot \left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t) \right]$$

pólos em: $s = \sigma \pm j\omega_d$ ou: $s = -\zeta\omega_n \pm j\omega_n \sqrt{1-\zeta^2}$

$$\%OS = \exp\left(-\zeta\pi/\sqrt{1-\zeta^2}\right) \times 100$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

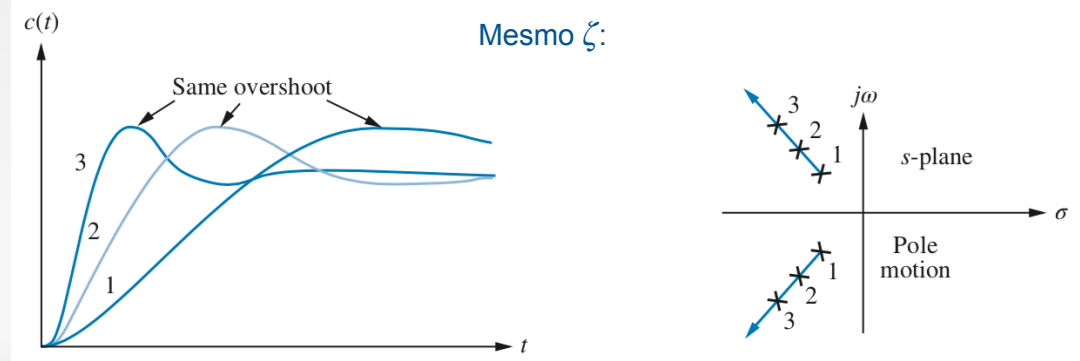
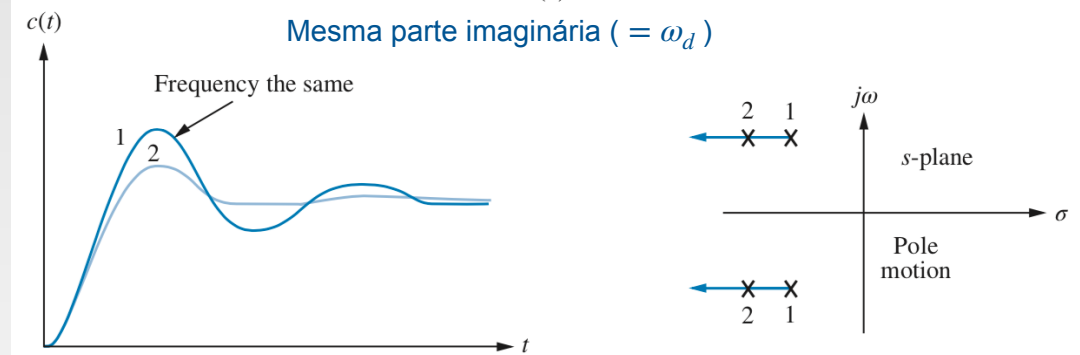
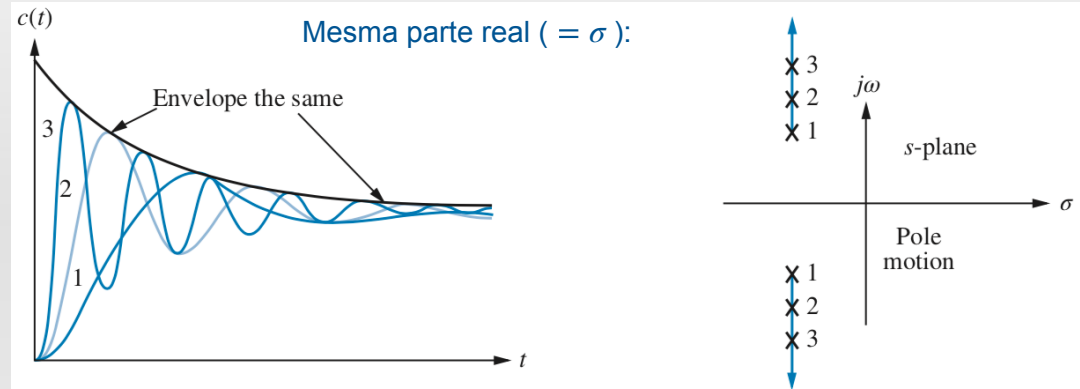
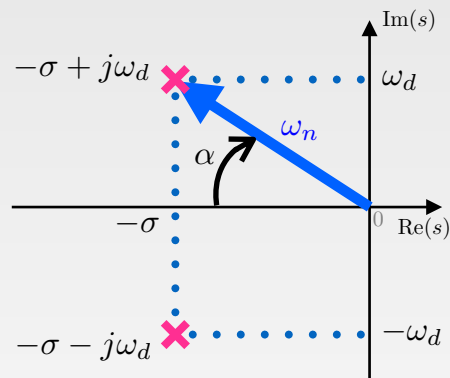
$$\zeta = \cos(\alpha)$$

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$t_s = \frac{-\ln(0,02\sqrt{1-\zeta^2})}{\zeta\omega_n}$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \quad \text{para: } 0 < \zeta < 0,9$$

$$t_r = \frac{0,6 + 2,16\zeta}{\omega_n}$$



Procedimento Para Projetos

- 1) Desenhe o diagrama do lugar das raízes para o sistema;
- 2) Assuma que é um sistema de 2ª ordem com ou sem zeros e encontre o ganho que cumpra com as especificações de resposta transitória;
- 3) Justifique a aproximação para um sistema de 2ª ordem encontrando a posição de todos os polos mais rápidos e considerando o fato de que estão muito mais afastados do eixo $j\omega$ que os pares dominantes de 2ª-orden!
- 4) Também verifique que os zeros de laço fechado são aproximadamente cancelados pelos polos de maior ordem (os dominantes).
- 5) Se esta hipótese não pode ser justificada, sua solução terá que ser simulada com o fim de garantir de que se cumpre com a especificação para a resposta transitória.