# Diagrama de Bode

Controle Automatico I
Prof. Fernando Passold
Nov-2009; Jul-2020; Jul-2022

# Introdução

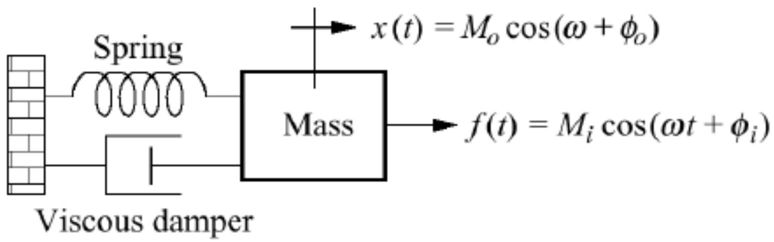
- Estabilidade e projeto da resposta transitória mediante ajuste de ganho:
  - Métodos baseados em resposta em frequência, diferentes do método baseado em RL, podem ser realizados sem a obrigatoriedade de una ferramenta computacional usando aproximações assintóticas.

- O projeto da resposta transitória mediante compensação em cascata:
  - Métodos baseados em resposta em frequência não são tão intuitivos como os baseados em RL.

- Projeto dos erros de estado estacionário mediante compensação en cascata:
  - Métodos baseados em resposta em frequência facilitam o projeto de compensadores derivativos de forma a acelerar a resposta do sistema ao mesmo tempo respeitando requerimentos de erros de regime permanente.

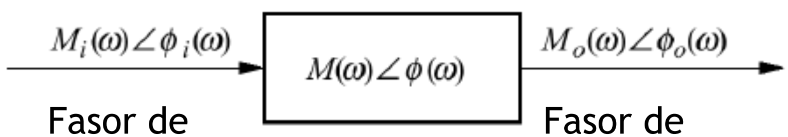
# Resposta em frequência: Definição…

- Ondas sinusoidais podem ser representadas como números complexos clamados fasores




(*a*)

*f(t)* = entrada de força, sinusoidal neste caso:




Fasor de entrada

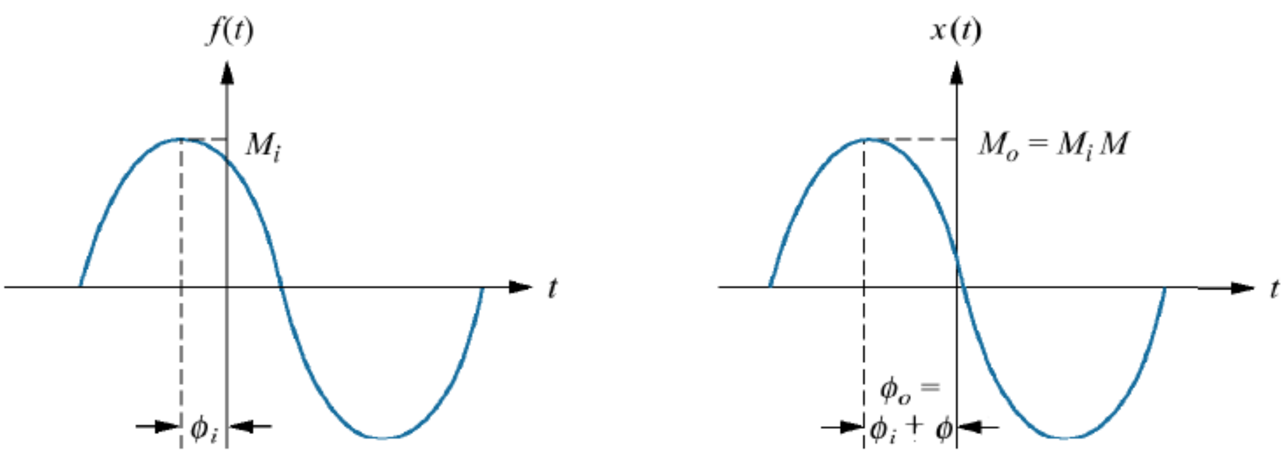
(*b*)

Fasor de saída

Definição:

$$M(w) = \frac{M_o(w)}{M_i(w)}$$

$$\phi(w) = \phi_o(w) - \phi_i(w)$$



# Definições Matemáticas...

Relações de Euler:

$e^{j\theta} = \cos\theta + j\sin\theta$

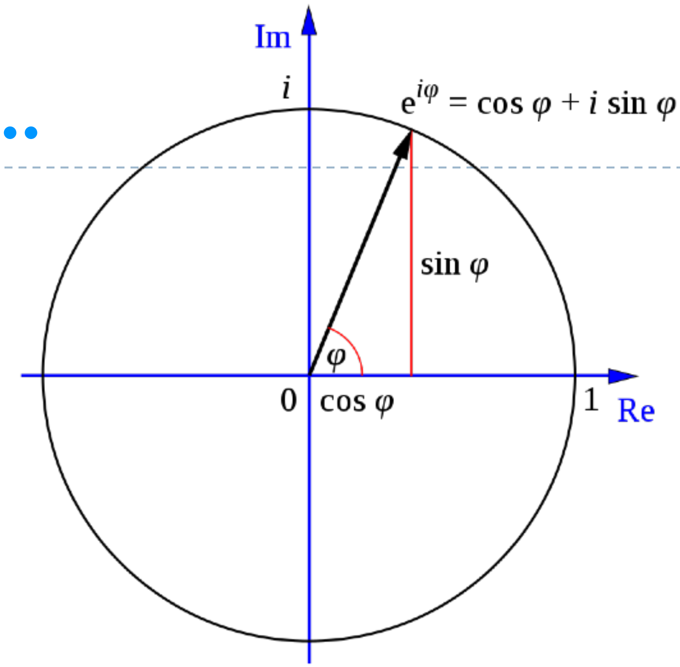
$z = x + jy = |z|\left[\cos(\varphi) + j\sin(\varphi)\right] = re^{j\varphi}$

$x$ = Parte real de $z$;

$y$ = Parte Imaginária de $z$;

$r = |z| = \sqrt{x^2 + y^2}$ = magnitude de $z$;

$\varphi$ = atan2$(y, x)$



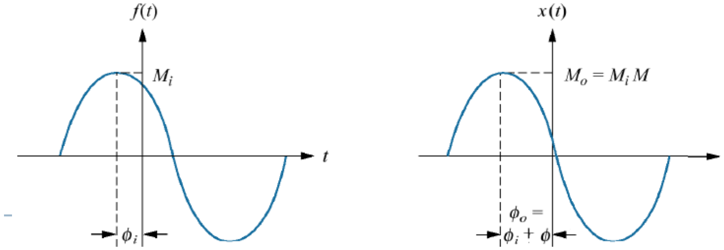

- Definição: $M(w)\angle\phi(w)$
  - Magnitude: $M(w) = M_o(w)/M_i(w)$
  - Fase: $\phi(w) = \phi_o(w) - \phi_i(w)$

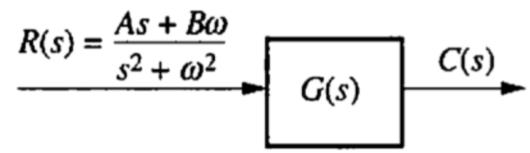
- Formatos de expressão:

  $r(t) = A\cos(wt) + B\sin(wt) = \sqrt{A^2 + B^2}\cos\left[wt - \tan^{-1}(B/A)\right]$

  1. Forma Polar: $M_i\angle\phi_i$
  2. Forma Retangular: $A - jB$
  3. Equação de Euler: $M_i e^{j\phi_i}$

# Expressões analíticas p/ resposta em frequência




Seja o sistema mostrado na fig. ao lado.
Este sistema é excitado por uma entrada sinusoidal:

$$r(t) = Acos(\omega t) + B\sin(\omega t)$$

$$r(t) = \sqrt{A^2 + B^2}\cos\left[\omega t - \tan^{-1}(B/A)\right]$$

$r(t) = M_i\angle\phi_i = M_i e^{j\phi_i}$, onde: $M_i = \sqrt{A^2 + B^2}$ e $\phi_i = -\tan^{-1}(B/A)$

A resposta forçada do sistema $G(s)$ para esta entrada resulta em:

$$C(s) = \frac{As + B\omega}{(s^2 + \omega^2)} \cdot G(s)$$

Expandindo usando frações parciais, teremos:

$C(s) = \dfrac{As + B\omega}{(s + j\omega)(s - j\omega)} = \dfrac{K_1}{s + j\omega} + \dfrac{K_2}{s - j\omega}$ + Termos frações parciais de $G(s)$

$$K_1 = \frac{As + B\omega}{s - j\omega}G(s)\bigg|_{s \to -j\omega} = \frac{1}{2}(A + jB)G(-j\omega) = \frac{1}{2}M_i e^{-j\phi_i} M_G e^{-j\phi_G}$$

$$K_1 = \frac{M_i M_G}{2}e^{-j(\phi_i + \phi_G)}$$

$$K_2 = \frac{As + B\omega}{s + j\omega}G(s)\bigg|_{s \to -j\omega} = \frac{1}{2}(A - jB)G(j\omega) = \frac{1}{2}M_i e^{j\phi_i} M_G e^{j\phi_G}$$

$$K_2 = \frac{M_i M_G}{2}e^{j(\phi_i + \phi_G)} = K_1^*$$

onde: $M_G = |G(j\omega)|$ e $\phi_G = \angle G(j\omega)$.

Então $C(s)$ resulta:

$$C(s) = \frac{\frac{M_i M_G}{2}e^{-j(\phi_i + \phi_G)}}{s + j\omega} + \frac{\frac{M_i M_G}{2}e^{j(\phi_i + \phi_G)}}{s - j\omega}$$

Cuja transformada inversa de Laplace rende:

$$c(t) = M_i M_G \left(\frac{e^{-j(\omega t + \phi_i + \phi_g)} + e^{j(\omega t + \phi_i + \phi_G)}}{2}\right)$$

$$c(t) = M_i M_G \cos(\omega t + \phi_i + \phi_G)$$

$$c(t) = M_o\angle\phi_o = (M_i\angle\phi_i)\cdot(M_G\angle\phi_G)$$

Note que:

$M_G\angle\phi_G =$ resposta em frequência.

Ou seja:

$$G(j\omega) = G(s)\Big|_{s \to j\omega}$$

# Exemplo 1: $G(s) = \dfrac{1}{s+2}$

Como: $s = jw$, para obter: $G(jw) = G(s)\big|_{s \to jw}$

$$G(j\omega) = \frac{1}{j\omega + 2} = \frac{1}{2 + j\omega}$$

$$G(j\omega) = \frac{1}{j\omega + 2} = \frac{1}{2 + j\omega} \cdot \frac{2 - j\omega}{2 - j\omega} = \frac{2 - j\omega}{4 + \omega^2}$$

Magnitude: $|G(j\omega)| = \dfrac{1}{|2 + j\omega|} = \dfrac{1}{\sqrt{2^2 + \omega^2}} = \dfrac{1}{\sqrt{\omega^2 + 4}}$

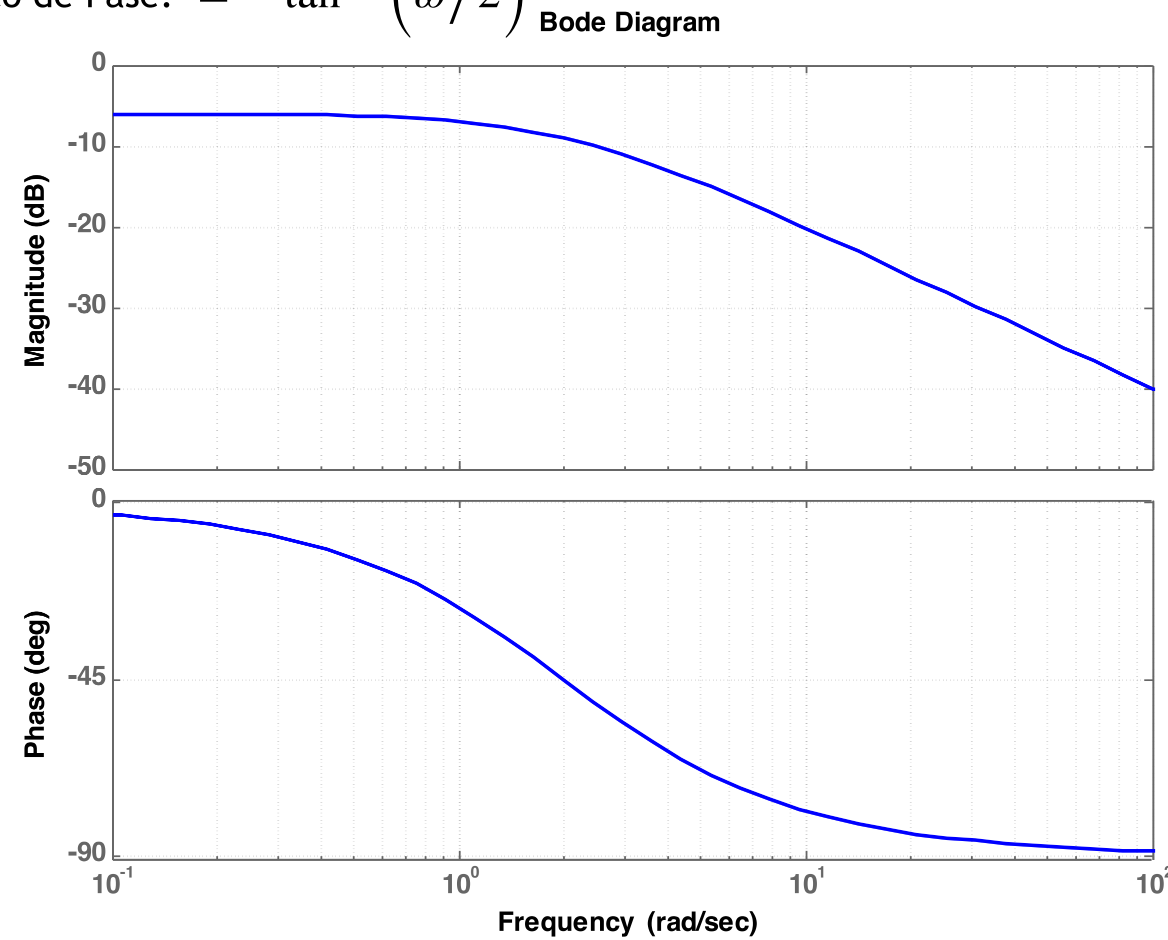
Magnitude(dB): $|G(j\omega)|\Big|_{dB} = 20\log\left(\dfrac{1}{\sqrt{\omega^2 + 4}}\right)$

Fase: $\angle G(j\omega) = \dfrac{1}{\tan^{-1}(\omega/2)} = -\tan^{-1}\left(\dfrac{\omega}{2}\right)$

# Exemplo 1: $G(s) = \dfrac{1}{s+2}$

Grafico de Magnitude: $= 20\log(1/\sqrt{\omega^2+4})$

Grafico de Fase: $= -\tan^{-1}\left(\omega/2\right)$




```
>> clear all
>> numg=1;
>> deng=([1 2]);
>> g=tf(numg,deng);
>> zpk(g)

Zero/pole/gain:
  1
-----
(s+2)

>> bode(g), grid
```

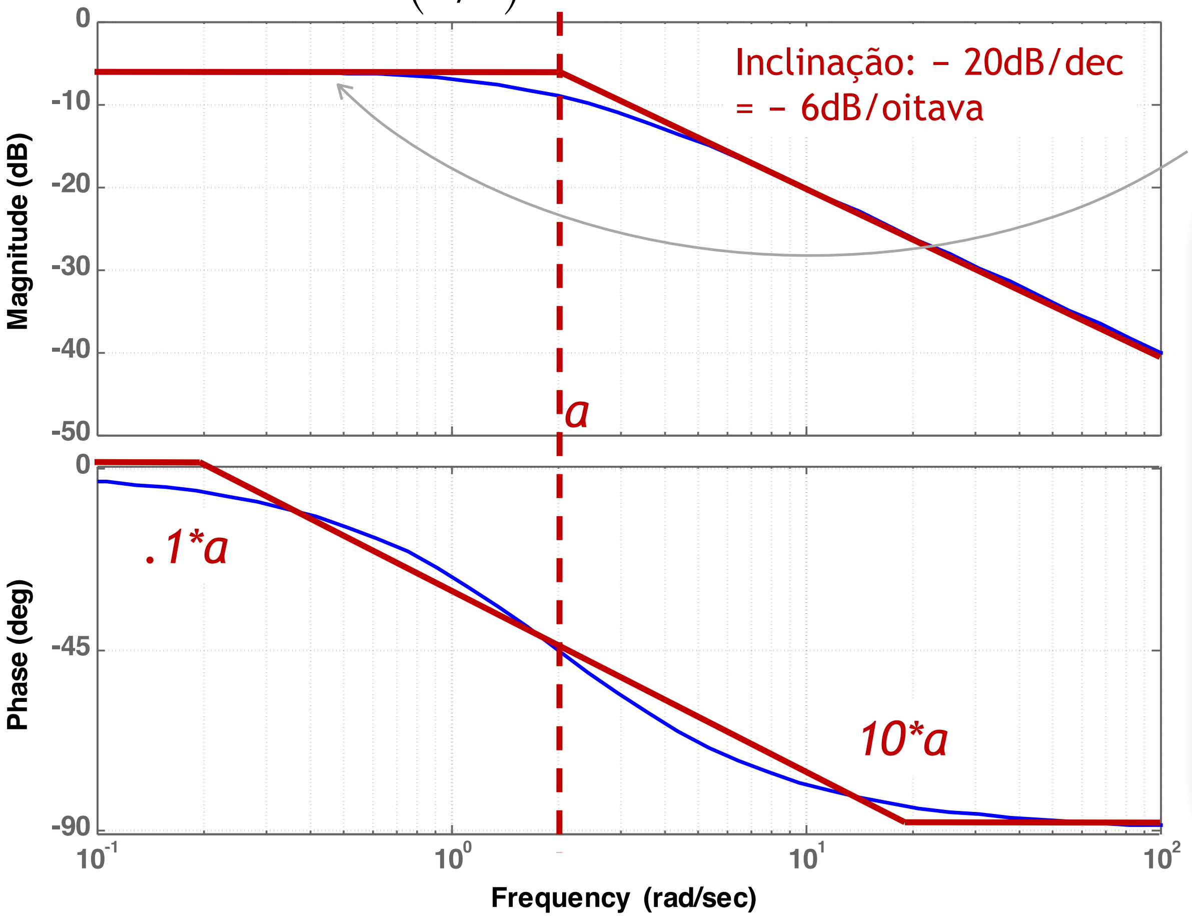
Ref: URL: http://www.swarthmore.edu/NatSci/echeeve1/Ref/LPSA/Bode/BodeHow.html

# Exemplo 1: $G(s) = \dfrac{1}{s+2}$

Grafico de Magnitude: $= 20\log(1/\sqrt{\omega^2+4})$ ou $= 20\log\left(1/\sqrt{\omega^2+a^2}\right)$

Grafico de Fase: $= -\tan^{-1}\left(\omega/2\right)$

**Bode Diagram**




**Assintóticamente:**

$$G(s) = \frac{1}{(s+a)} = \frac{1}{a\left(\frac{s}{a}+1\right)}$$

Para baixas freq. (*jw<a; suponha w=a/10*):

$$|G(j\omega)| = 20\log\left(\frac{1}{\sqrt{\left(\frac{a}{10}\right)^2+a^2}}\right) = 20\log\left(\frac{1}{\sqrt{\frac{101a^2}{100}}}\right)$$

$$|G(j\omega)| \cong 20\log\left(\frac{1}{a}\right) = -20\log(a)$$

$$|G(j\omega)| = -20\log(2) = -6,0205999133 \text{ dB}$$

Lembrando que:

$$\log_a b = x \quad \therefore \quad a^x = b$$

$$\log(1) = 0$$

$$\log(2) = 0{,}3010299957$$

$$\log(10) = 1$$

$$\log(100) = 2 \quad \therefore \quad 10^2 = 100$$

$$\log\left(1/10\right) = -\log(10) = -1$$

$$\log(a \cdot b) = \log(a) + \log(b)$$

$$\log\left(a/b\right) = \log(a) - \log(b)$$

Ref: URL: http://www.swarthmore.edu/NatSci/echeeve1/Ref/LPSA/Bode/BodeHow.html

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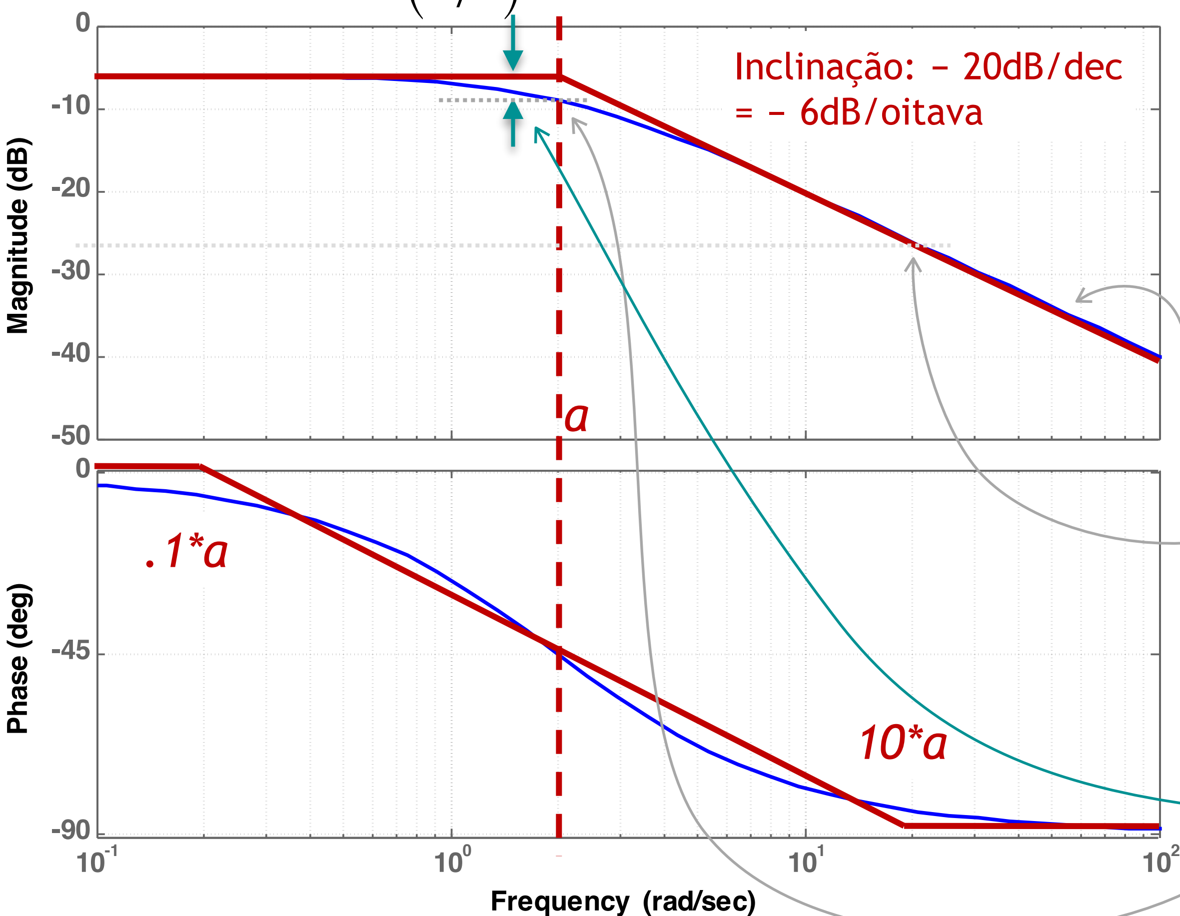


## Assintóticamente:

$$G(s) = \frac{1}{(s+a)} = \frac{1}{a\left(\frac{s}{a}+1\right)}$$

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Para altas freq. (*jw>a; suponha w=10a*):

$$|G(j\omega)| = 20\log\left(\frac{1}{\sqrt{(10\,a)^2+a^2}}\right) = 20\log\left(\frac{1}{\sqrt{101a^2}}\right)$$

$$|G(j\omega)| \cong 20\log\left(\frac{1}{\sqrt{100a}}\right) \cong \underbrace{20\log(1)}_{0} - 20\log(10a)$$

$$|G(j\omega)| \cong -20\log(\omega)$$

$$|G(j\omega)|\Big|_{\omega=10a} \cong -20\log(2\cdot 10) = -26,0205999133 \text{ dB}$$

E para $\omega = a$, teremos:

$$|G(j\omega)| = 20\log\left(\frac{1}{\sqrt{a^2+a^2}}\right)$$

$$|G(j\omega)| = -20\log(\sqrt{2a^2}) = \underbrace{-20\log(\sqrt{2})}_{-3,0103 \text{ dB}} - 20\log a$$

$$\underbrace{-20\log(\sqrt{2})}_{-3,0103 \text{ dB}} - 20\log 2 = -9,0309 \text{ dB}$$

Ref: URL: http://www.swarthmore.edu/NatSci/echeeve1/Ref/LPSA/Bode/BodeHow.html

Para altas freq. (*jw>a*):


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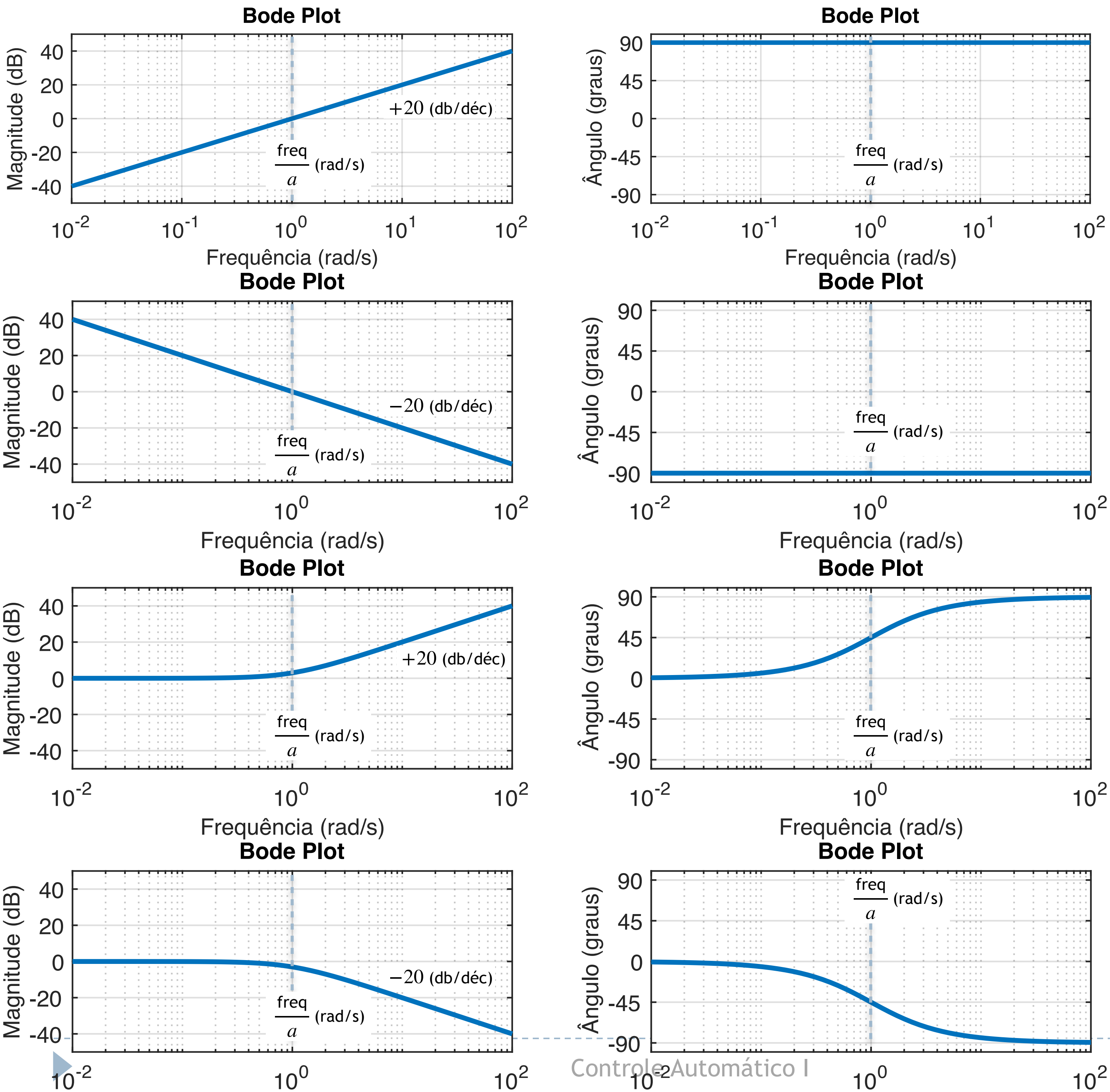



a) $G(s) = s$
(Derivador Puro);

b) $G(s) = \dfrac{1}{s}$
(Integrador Puro);

c) $G(s) = (s + a)$

d) $G(s) = \dfrac{1}{(s + a)}$

# bode_lado_lado.m

```
% Mostrar diagrama de Bode: Magnitude x Fase, lado a lado
% Entrada: variável G = tf(.)
% Fernando Passold, em 10.06.2022

W={0.01,100};
[MAG,PHASE,Wb] = bode(G,W);

% Diagrama de Magnitude
subplot(121);
% bodemag(G,W); % até poderia ser usado
MAG=squeeze(MAG); % reduce dimensions
mag = 20*log10(MAG);
semilogx(Wb,mag,'LineWidth',3)
grid on
axis([0.01 100 -50 50])
title ('Bode Plot')
xlabel('Frequência (rad/s)');
ylabel('Magnitude (dB)');
yticks([-40 -20 0 20 40]);

% Diagrama de Fase
subplot(122);
% Detalhe: PHASE = 1 x 1 x 41 !!!
PHASE=squeeze(PHASE); % reduce dimensions
semilogx(Wb,PHASE,'LineWidth',3)
axis([0.01 100 -100 100])
grid on
title ('Bode Plot')
xlabel('Frequência (rad/s)');
ylabel('Ângulo (graus)');
yticks([-90 -45 0 45 90]);
```

# Revisão de traçados de Diagramas de Bode…

- Diagrama de Bode para:

$$G(s) = (s + a)$$

  - Se $s = jw$: (Derivador Puro)

$$G(jw) = (jw + a) = a\left(j\frac{w}{a} + 1\right)$$

  - Para baixas frequências ($w < a$):

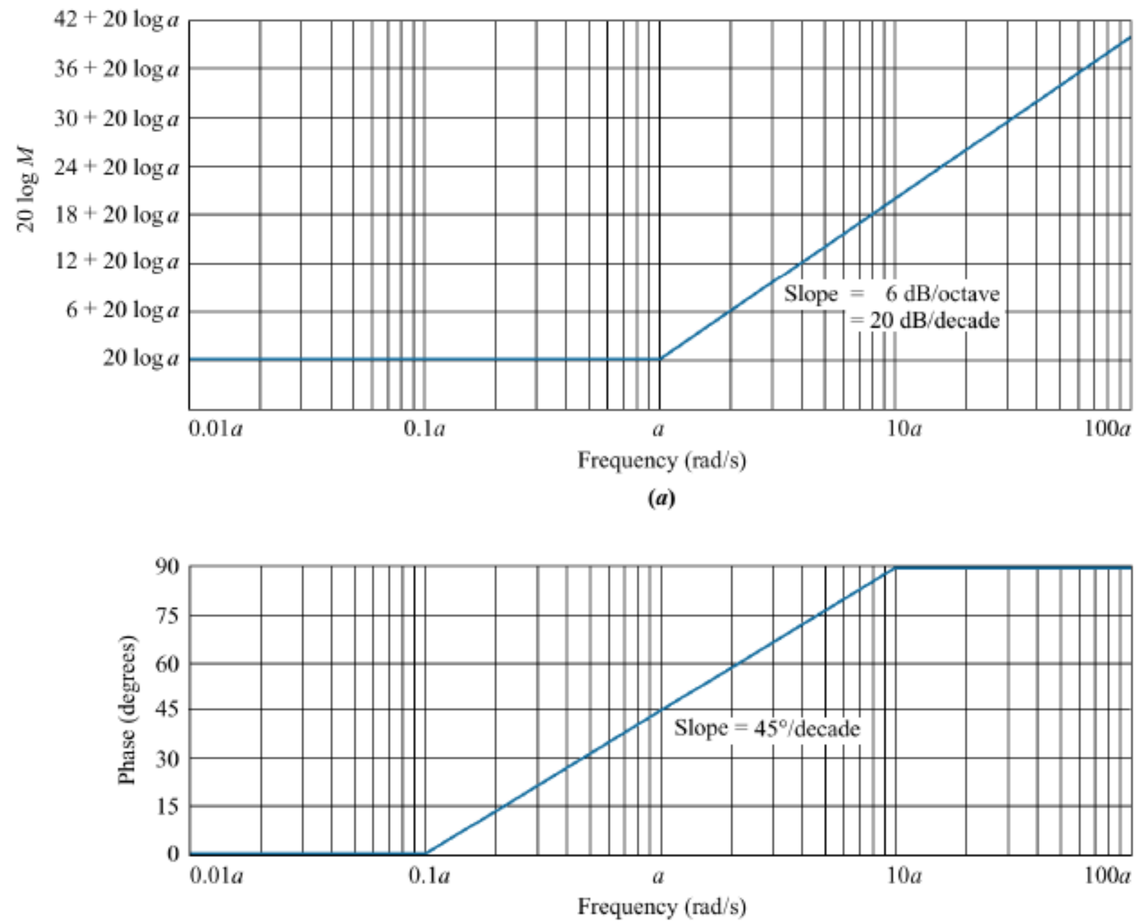
$$G(jw) \approx a$$

$$20\log(M) = 20\log(a)$$

  - Para frequências elevadas ($w > a$)

$$G(jw) \approx a$$

$$20\log(M) = 20\log(a)$$



# Exemplo 2) $G(s) = K \cdot \dfrac{(s+10)}{(s+1)(s+100)}$

1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

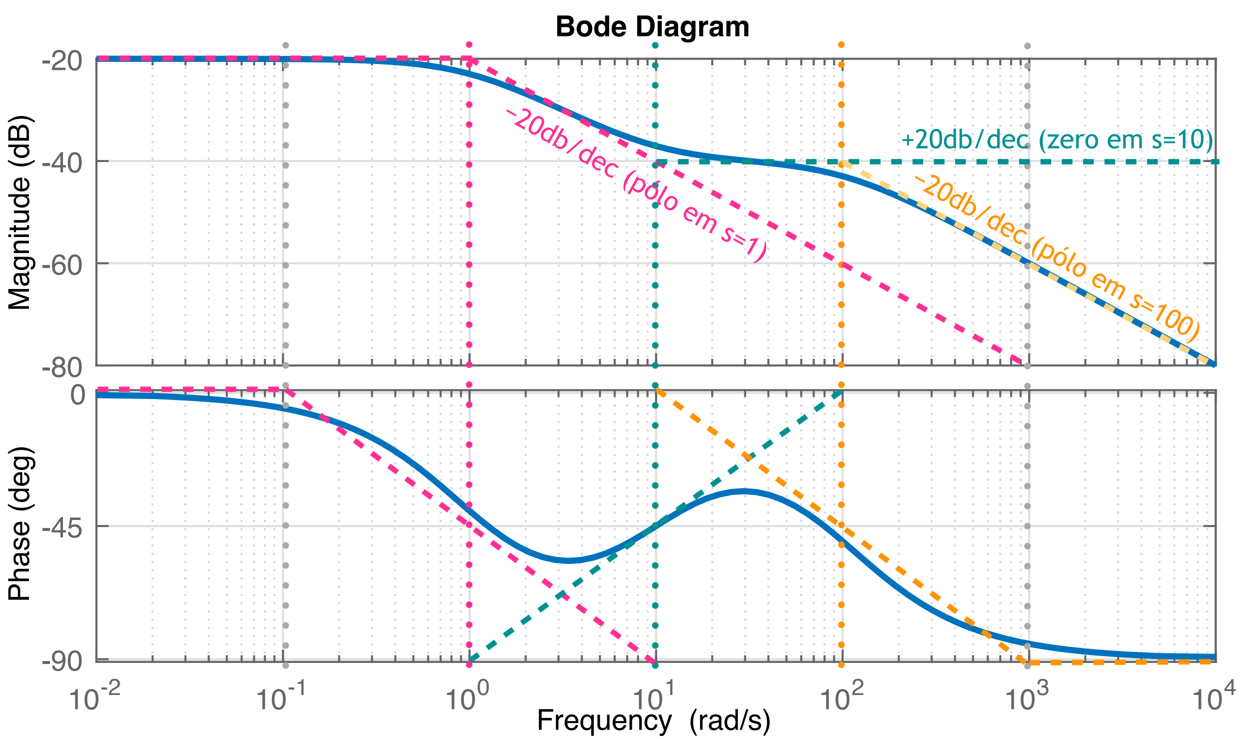
$$\lim_{t\to\infty} g(t) = \lim_{s\to 0} s \cdot G(s)$$

Neste caso: Ganho DC quando aplicado degrau

$$= K \cdot \lim_{s\to 0} \underbrace{\left(\frac{1}{\not{s}}\right)}_{\text{Degrau}} \cdot \left[\frac{\not{s}\,(s+10)}{(s+1)(s+10)}\right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0{,}1$$

$$\text{Ganho DC}\Big|_{\text{dB}} = 20\log(K \cdot 1/10) = -20 + 20\log(K)\ \text{dB}$$

2. Lembrar que:
   cada pólo decrementa ganho de: -20 db/déc;
   cada zero incrementa ganho de: +20 db/déc.




```
>> G=tf([1 10],poly([-1 -100]))
G =
        s + 10
   ----------------
   s^2 + 101 s + 100

>> zpk(G)

      (s+10)
   ------------
   (s+100) (s+1)

>> bode(G)
>> grid
```

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Neste caso: Ganho DC quando aplicado degrau

$$= K \cdot \lim_{s\to 0} \underbrace{\left(\frac{1}{\not s}\right)}_{\text{Degrau}} \cdot \left[\frac{\not s\,(s+10)}{(s+1)(s+10)}\right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0{,}1$$

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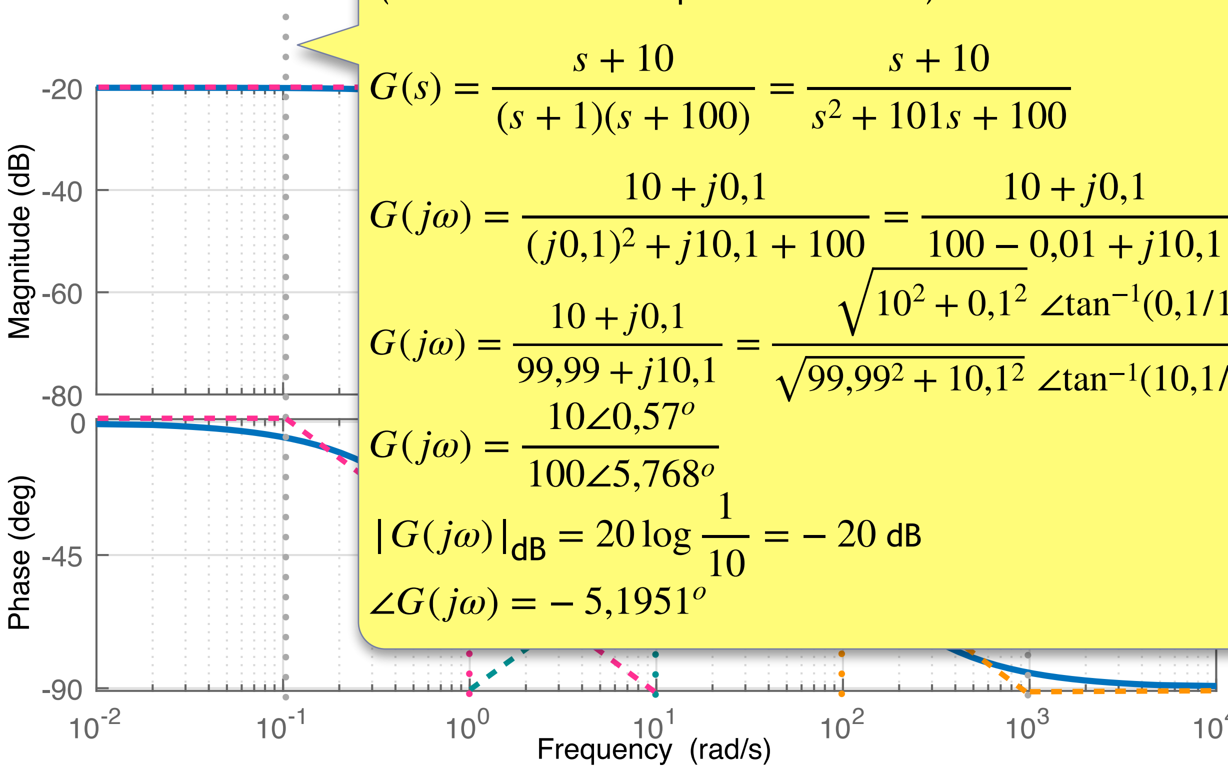
de: -20 db/déc;
de: +20 db/déc.




Valores em $\omega = 0{,}1$ rad/s
(1 década abaixo do pólo em $s = -1$):

$$G(s) = \frac{s+10}{(s+1)(s+100)} = \frac{s+10}{s^2 + 101s + 100}$$

$$G(j\omega) = \frac{10 + j0{,}1}{(j0{,}1)^2 + j10{,}1 + 100} = \frac{10 + j0{,}1}{100 - 0{,}01 + j10{,}1}$$

$$G(j\omega) = \frac{10 + j0{,}1}{99{,}99 + j10{,}1} = \frac{\sqrt{10^2 + 0{,}1^2}\ \angle\tan^{-1}(0{,}1/10)}{\sqrt{99{,}99^2 + 10{,}1^2}\ \angle\tan^{-1}(10{,}1/99{,}99)}$$

$$G(j\omega) = \frac{10\angle 0{,}57^o}{100\angle 5{,}768^o}$$

$$|G(j\omega)|_{\text{dB}} = 20\log\frac{1}{10} = -20\ \text{dB}$$

$$\angle G(j\omega) = -5{,}1951^o$$

```
>> G=tf([1 10],poly([-1 -100]))
G =
      s + 10
  ----------------
  s^2 + 101 s + 100

>> zpk(G)

    (s+10)
  ------------
  (s+100) (s+1)

>> bode(G)
>> grid
```

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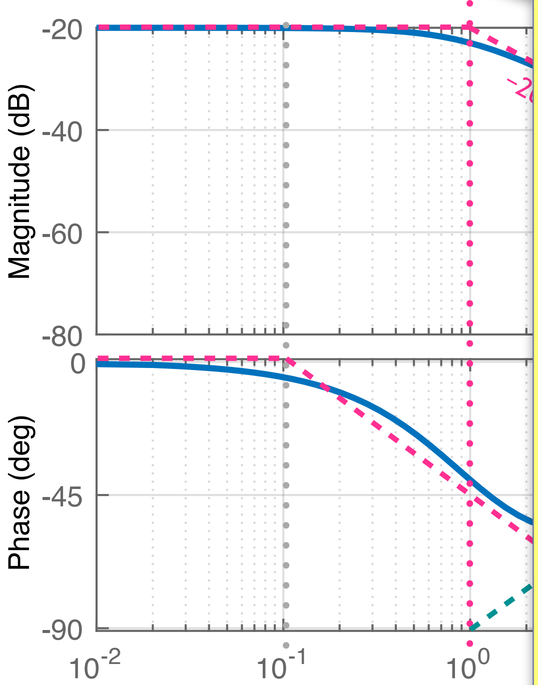
déc;
déc.




0],poly([-1 -100]))
0
-------
s + 100
---
+1)

Valores em $\omega = 1$ rad/s (no pólo em $s = -1$):

$$G(s) = \frac{s+10}{(s+1)(s+100)} = \frac{s+10}{s^2+101s+100}$$

$$G(j\omega) = \frac{10+j1}{(j)^2+j101+100} = \frac{10+j1}{100-1+j101}$$

$$G(j\omega) = \frac{10+j1}{99+j101} = \frac{\sqrt{10^2+1^2}\ \angle\tan^{-1}(1/10)}{\sqrt{99^2+101^2}\ \angle\tan^{-1}(101/99)}$$

$$G(j\omega) = \frac{\sqrt{101}\ \angle 5.7106^o}{\underbrace{\sqrt{20002}}_{\sqrt{100^2+100^2}=\sqrt{20000}}\qquad \angle 45.573^o}$$

$$G(j\omega) = \frac{10{,}05\angle 5{,}7106^o}{141{,}43\angle 45.573^o} = 0.07106\ \angle -39.862^o$$

$$|G(j\omega)|_{\text{dB}} = 20\log(0{,}07106) = -22{,}968\ \text{dB}$$

$$\angle G(j\omega) = -39{,}862^o$$

# Exemplo 2) $G(s) = K \cdot \dfrac{(s+10)}{(s+1)(s+100)}$

1. Cálculo do ganho “DC" (ganho de $G(s)$ em regime permanente):

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Neste caso: Ganho DC quando aplicado degrau:

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$$\text{Ganho DC}\Big| = 20\log(K \cdot 1/10) = -20 + 20\log(K) \text{ dB}$$

Valores em $\omega = 10$ rad/s (no pólo em $s = -10$):

$$G(s) = \frac{s+10}{(s+1)(s+100)} = \frac{s+10}{s^2+101s+100}$$

$$G(j\omega) = \frac{10+j10}{(j10)^2 + j1010 + 100} = \frac{10+j10}{\cancel{100} - \cancel{100} + j1010}$$

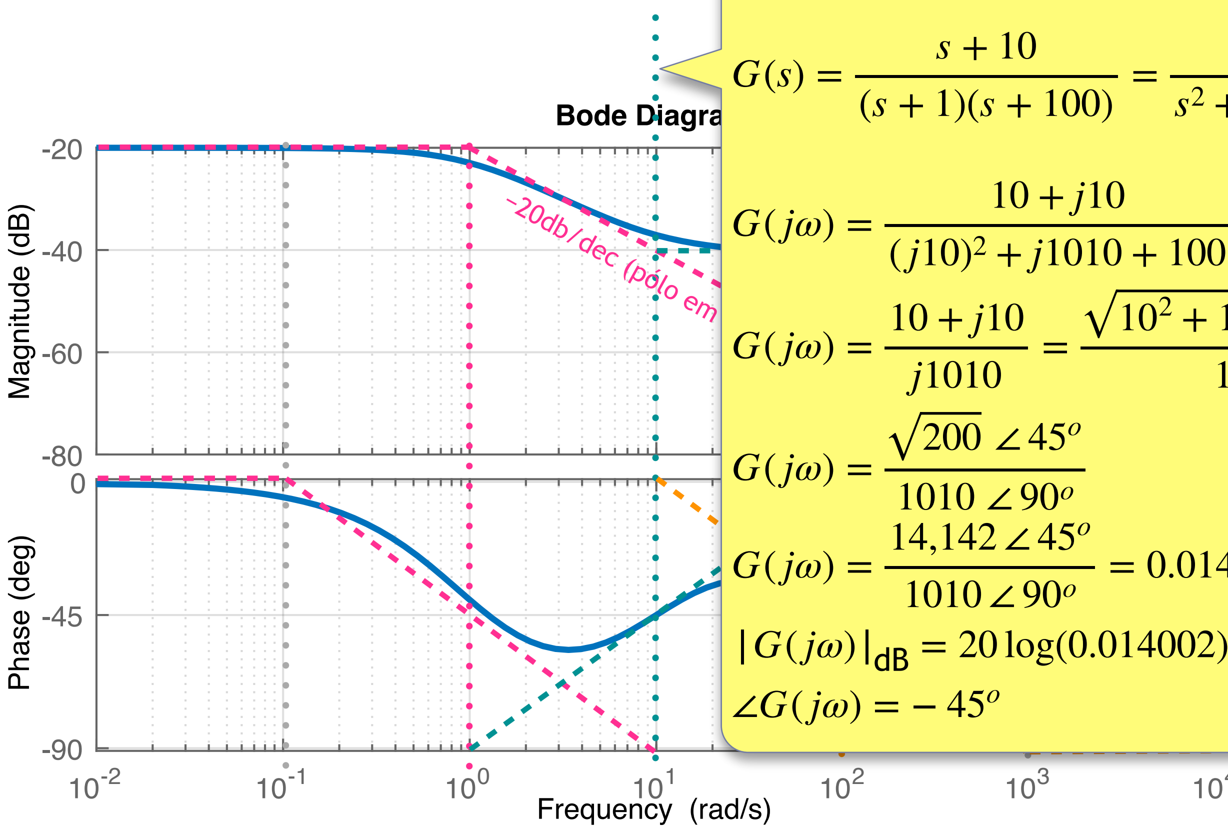
$$G(j\omega) = \frac{10+j10}{j1010} = \frac{\sqrt{10^2+10^2}\ \angle \tan^{-1}(10/10)}{1010\ \angle 90^o}$$

$$G(j\omega) = \frac{\sqrt{200}\ \angle 45^o}{1010\ \angle 90^o}$$

$$G(j\omega) = \frac{14{,}142\ \angle 45^o}{1010\ \angle 90^o} = 0.014002\ \angle -45^o$$

$$|G(j\omega)|_{\text{dB}} = 20\log(0.014002) = -37{,}076 \text{ dB}$$

$$\angle G(j\omega) = -45^o$$




```
100]))
>> bode(G)
>> grid
```

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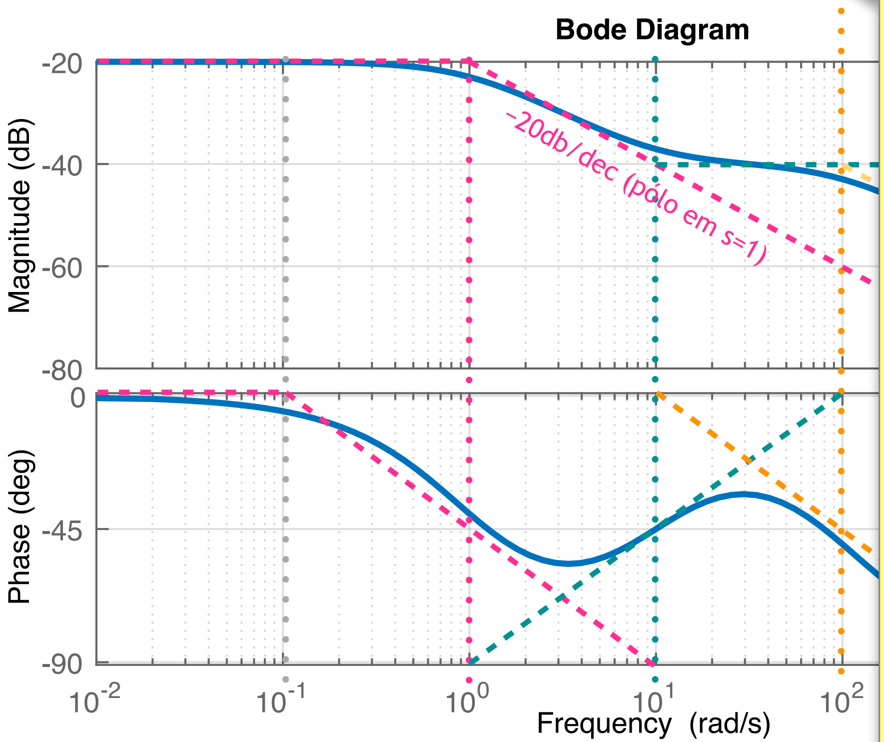



Valores em $\omega = 100$ rad/s (no pólo em $s = -100$):

$$G(s) = \frac{s+10}{(s+1)(s+100)} = \frac{s+10}{s^2+101s+100}$$

$$G(j\omega) = \frac{10+j100}{(j100)^2+j10100+100} = \frac{10+j100}{100-10000+j10100}$$

$$G(j\omega) = \frac{10+j100}{-9900+j10100} = \frac{\sqrt{10^2+100^2}\ \angle\tan^{-1}(100/10)}{\sqrt{(-9900)^2+10100^2}\ \angle\tan^{-1}(10100/-9900)}$$

$$G(j\omega) = \frac{\sqrt{10100}\ \angle 84{,}289^o}{\sqrt{200020000}\ \angle 134{,}43^o}$$

$$G(j\omega) = \frac{100{,}5\angle 84{,}289^o}{14143\angle 134.43^o} = 0{,}007106\ \angle -50.138^o$$

$$|G(j\omega)|_{\text{dB}} = 20\log(0{,}007106) = -42.968\ \text{dB}$$

$$\angle G(j\omega) = -50{,}138^o$$

# Exemplo 2) $G(s) = K \cdot \dfrac{(s+10)}{(s+1)(s+100)}$

1. Cálculo do ganho “DC" (ganho de $G(s)$ em regime permanente):

$\lim_{t\to\infty} g(t) = \lim_{s\to 0} s \cdot G(s)$ ...

Nes...

$$= K \cdot \underbrace{\lim_{s\to 0} \left(\frac{1}{\not s}\right)}_{\text{Degrau}} \cdot \left[\frac{\not s\,(s+10)}{(s+1)(s+10)}\right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0{,}1$$

... DC $\Big|_{dB} = 20\log(K \cdot 1/10) = -20 + 20\log(K)$ dB

...rar que:

...pólo decrementa ganho de: -20 db/déc;

...zero incrementa ganho de: +20 db/déc.

Valores em $\omega = 1000$ rad/s

(1 década acima do pólo em $s = -100$):

$$G(s) = \frac{s+10}{(s+1)(s+100)} = \frac{s+10}{s^2+101s+100}$$

$$G(j\omega) = \frac{10+j1000}{(j1000)^2 + j10100 + 100} = \frac{10+j1000}{100 - 1000000 + j10100}$$

$$G(j\omega) = \frac{10+j1000}{-999900 + j10100}$$

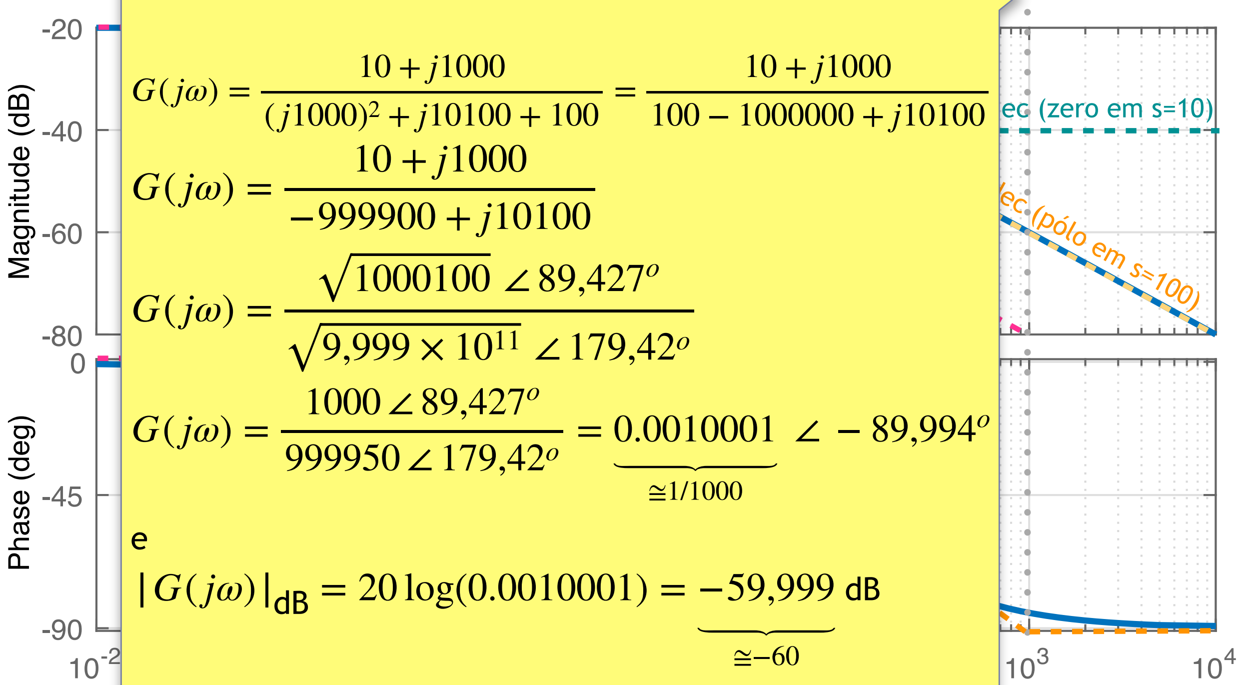
$$G(j\omega) = \frac{\sqrt{1000100}\ \angle 89{,}427^o}{\sqrt{9{,}999\times 10^{11}}\ \angle 179{,}42^o}$$

$$G(j\omega) = \frac{1000\angle 89{,}427^o}{999950\angle 179{,}42^o} = \underbrace{0.0010001}_{\cong 1/1000}\ \angle -89{,}994^o$$

e

$$|G(j\omega)|_{dB} = 20\log(0.0010001) = \underbrace{-59{,}999}_{\cong -60} \text{ dB}$$

$$\angle G(j\omega) \cong -90^o$$




```
>> G=tf([1 10],poly([-1 -100]))
G =
        s + 10
   -----------------
   s^2 + 101 s + 100

>> zpk(G)

     (s+10)
   -------------
   (s+100) (s+1)

>> bode(G)
>> grid
```

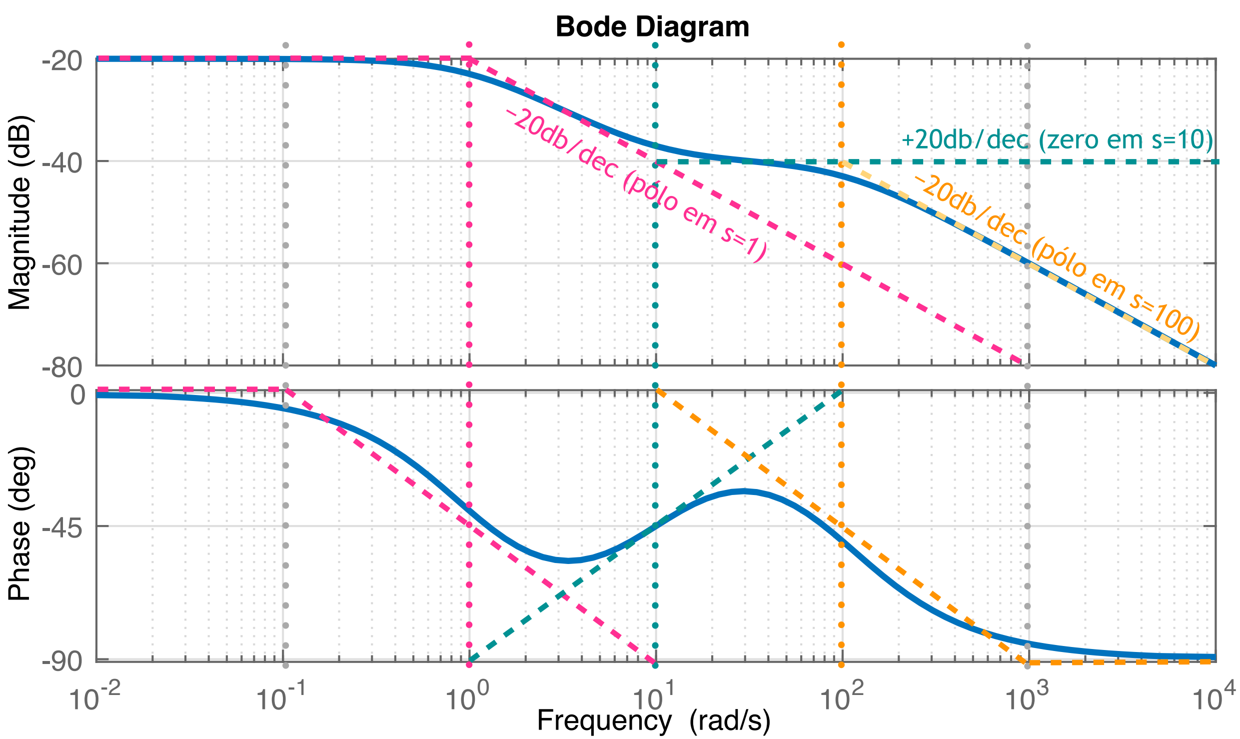
# Exemplo 2) $G(s) = K \cdot \dfrac{(s+10)}{(s+1)(s+100)}$

1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):
   $\lim_{t\to\infty} g(t) = \lim_{s\to 0} s \cdot G(s)$
   Neste caso: Ganho DC quando aplicado degrau

$$= K \cdot \lim_{s\to 0} \underbrace{\left(\frac{1}{\not s}\right)}_{\text{Degrau}} \cdot \left[\frac{\not s\,(s+10)}{(s+1)(s+10)}\right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0{,}1$$

$$\text{Ganho DC}\Big|_{\text{dB}} = 20\log(K \cdot 1/10) = -20 + 20\log(K)\ \text{dB}$$

2. Lembrar que:
   cada pólo decrementa ganho de: -20 db/déc;
   cada zero incrementa ganho de: +20 db/déc.



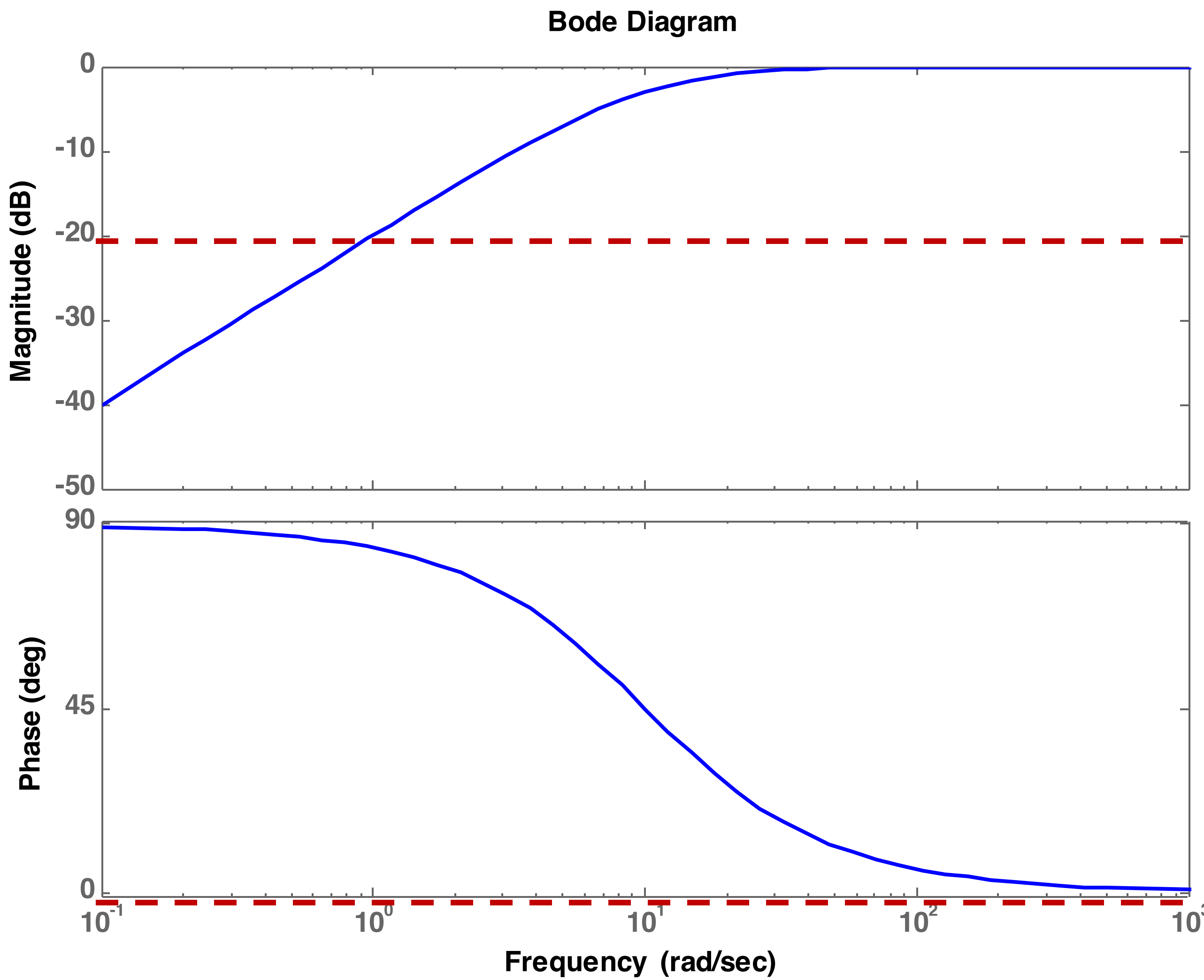

```
>> G=tf([1 10],poly([-1 -100]))
G =
        s + 10
  ----------------
  s^2 + 101 s + 100

>> zpk(G)

     (s+10)
  ------------
  (s+100) (s+1)

>> bode(G)
>> grid
```

# Outros exemplos:

$$G_1(s) = \frac{s}{(s+10)}$$

$$G_1(s) = \frac{1}{10} \cdot \frac{s}{\left(\frac{s}{10}+1\right)}$$

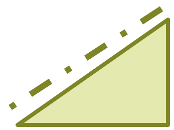
Linha de base, ganho = 0,1 (-20 dB)




$$\log(10) = 1$$

$$\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$$

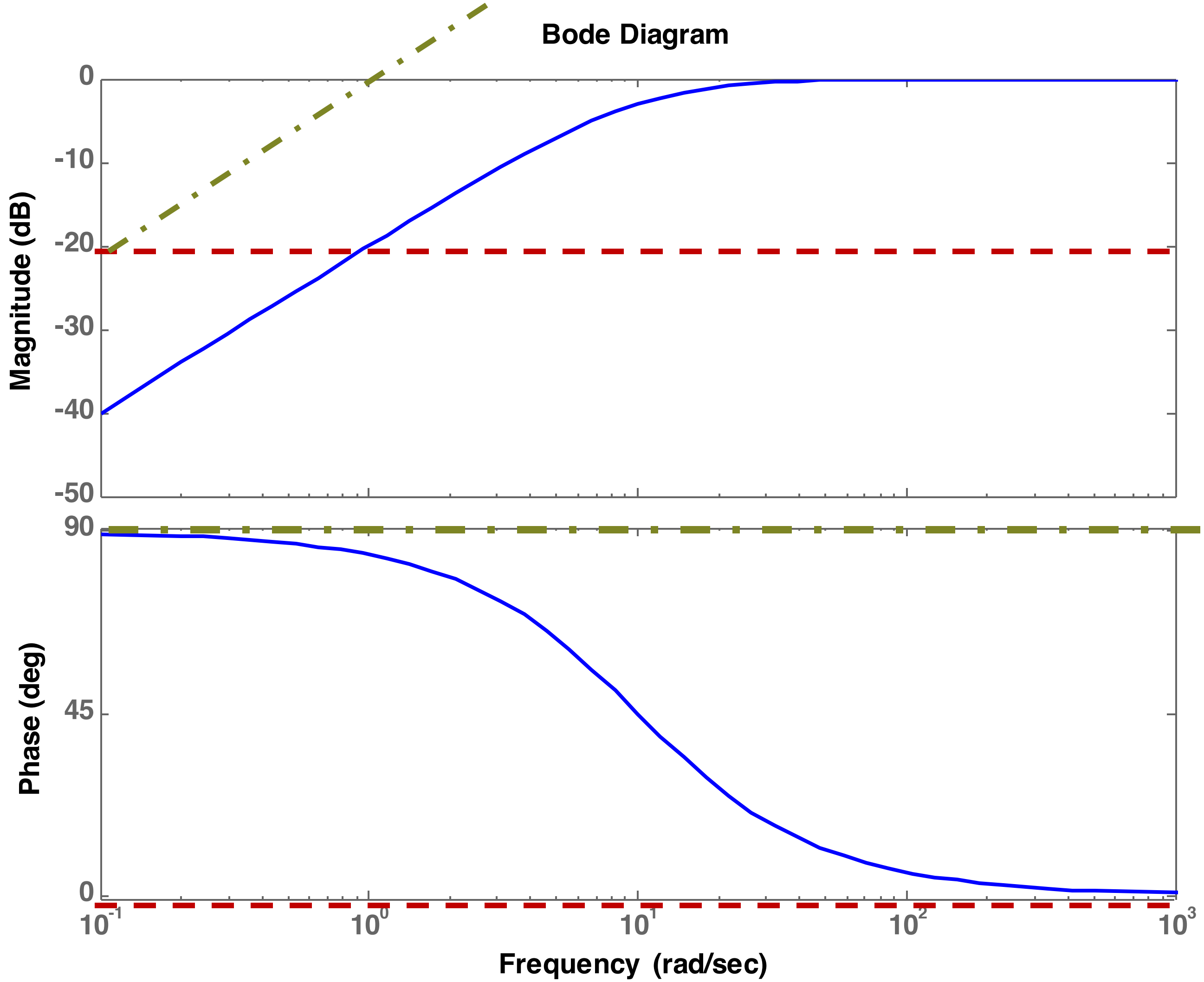
# Outros exemplos:




$$G_1(s) = \frac{s}{(s+10)}$$

$$G_1(s) = \frac{1}{10} \cdot \frac{s}{\left(\frac{s}{10}+1\right)}$$

Zero na origem




$$\log(10) = 1$$

$$\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$$

# Outros exemplos:

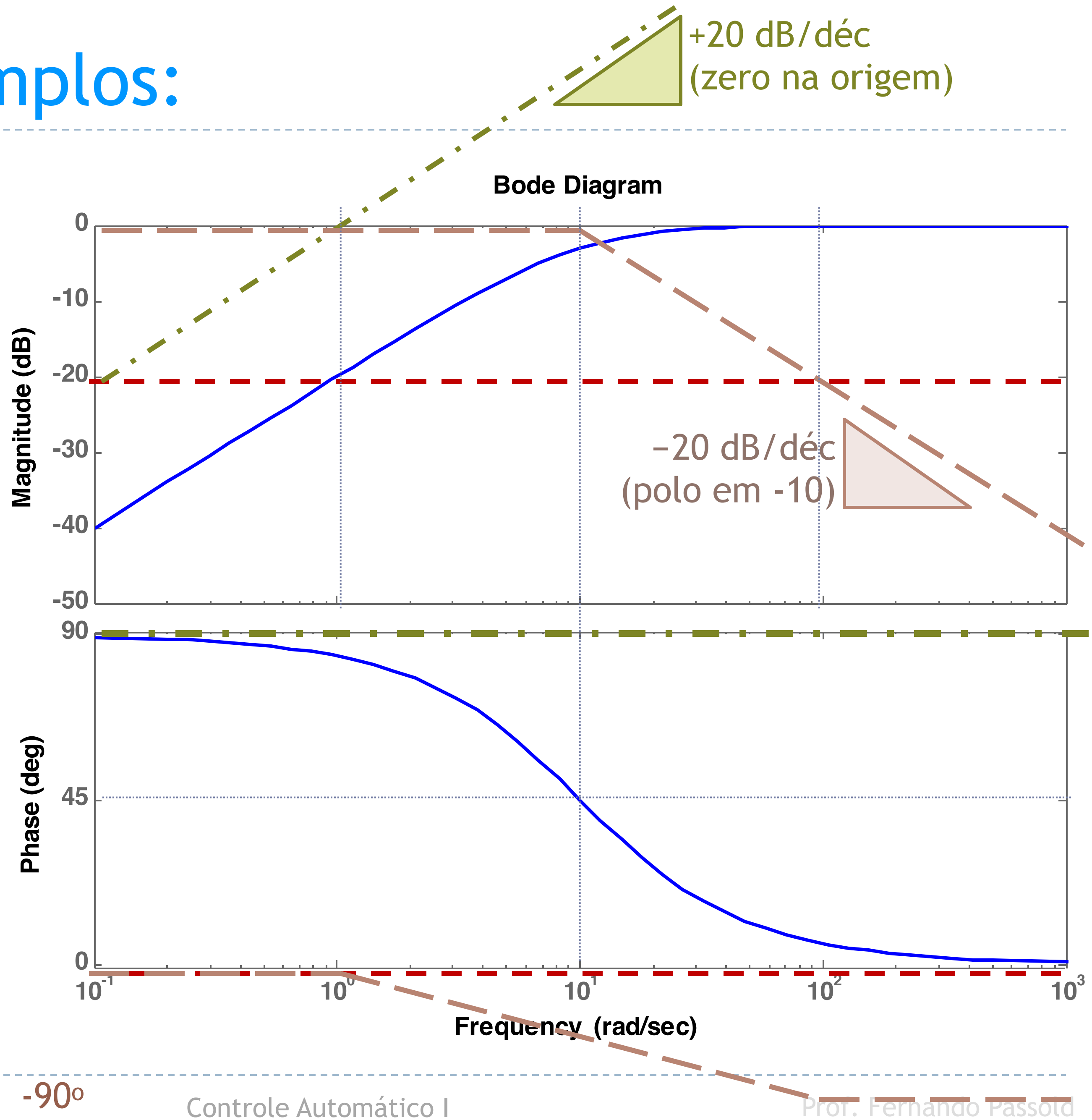
$$G_1(s) = \frac{s}{(s+10)}$$

$$G_1(s) = \frac{1}{10} \cdot \frac{s}{\left(\frac{s}{10}+1\right)}$$

Polo real em $s = -10$

$$\log(10) = 1$$

$$\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$$



# Outros exemplos:

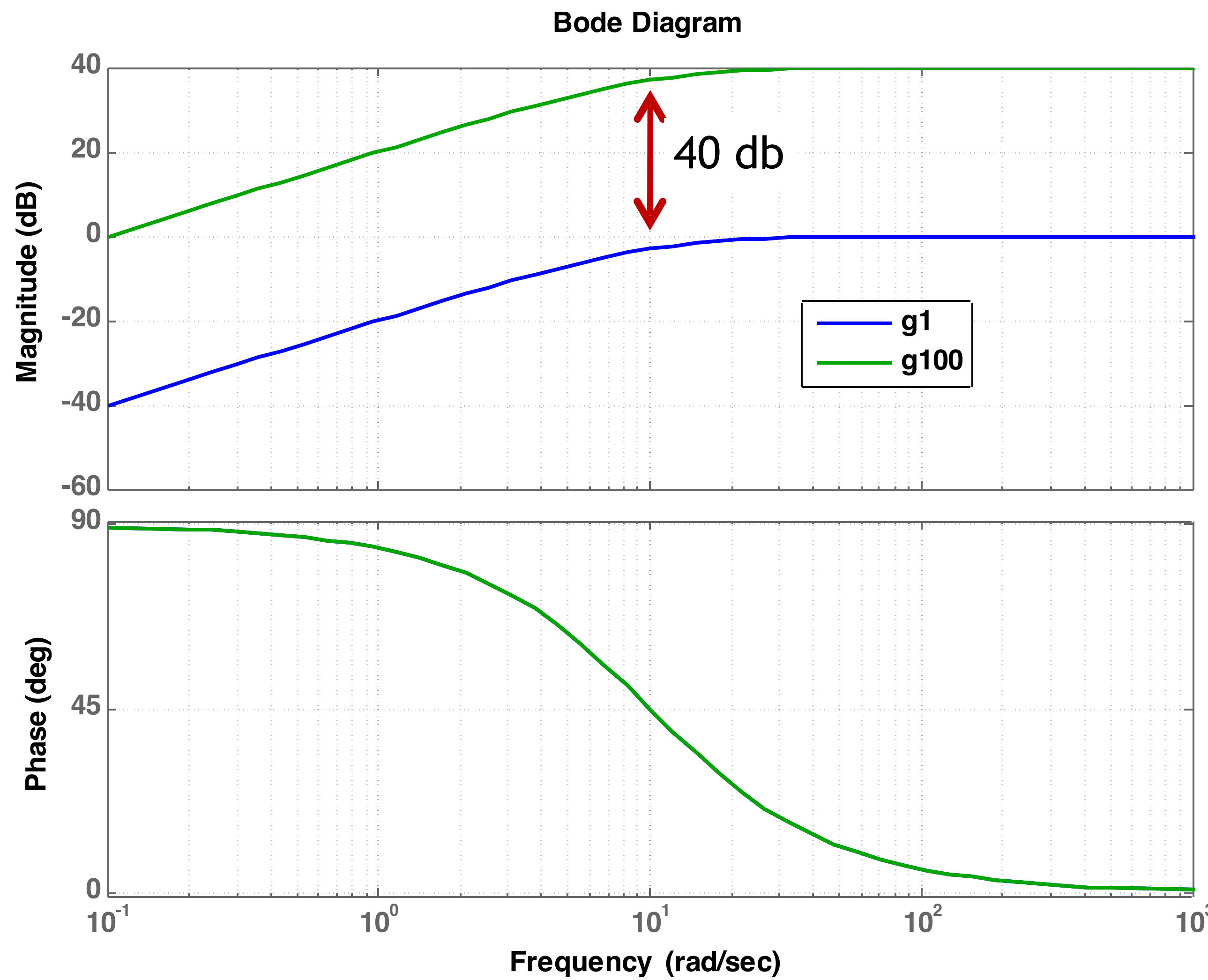
$$G_1(s) = \frac{s}{(s+10)}$$

$$G_{100}(s) = 100\frac{s}{(s+10)}$$

$$G_{100}(s) = 10\frac{s}{s+\frac{s}{10}}$$

Diferença:
Ganho de magnitude:

$$\log(100) = 2$$

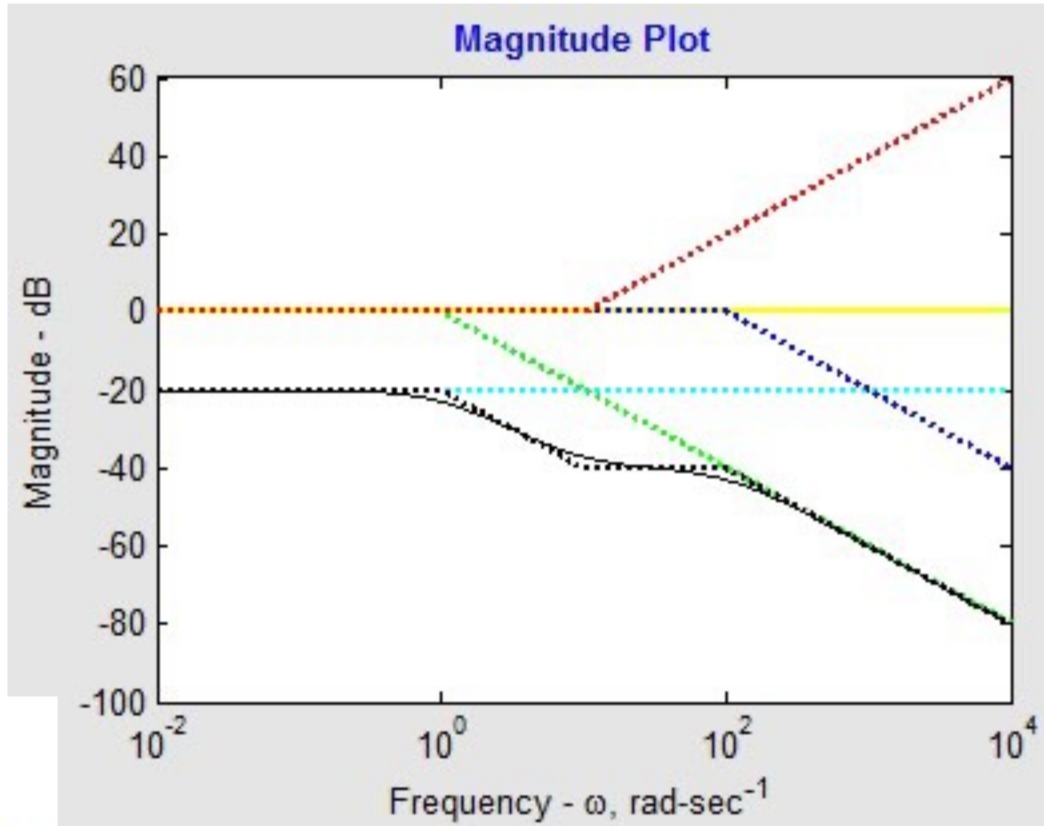


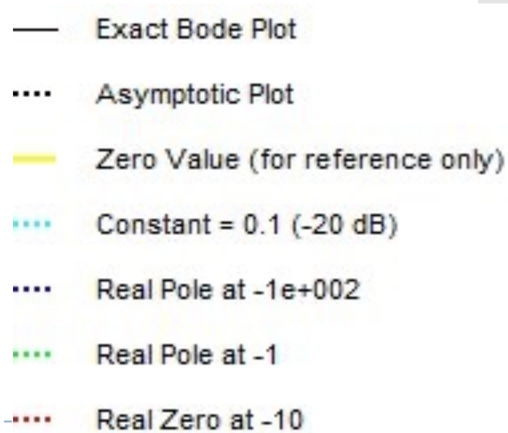
# Outros exemplos:

- $$G_2(s) = \frac{s+10}{(s+1)(s+100)}$$

$$= \frac{10\left(\frac{s}{10}+1\right)}{(s+1)\cdot 100\left(\frac{s}{100}+1\right)}$$

$$= \frac{10}{100}\cdot\frac{\left(\frac{s}{10}+1\right)}{(s+1)\left(\frac{s}{100}+1\right)}$$



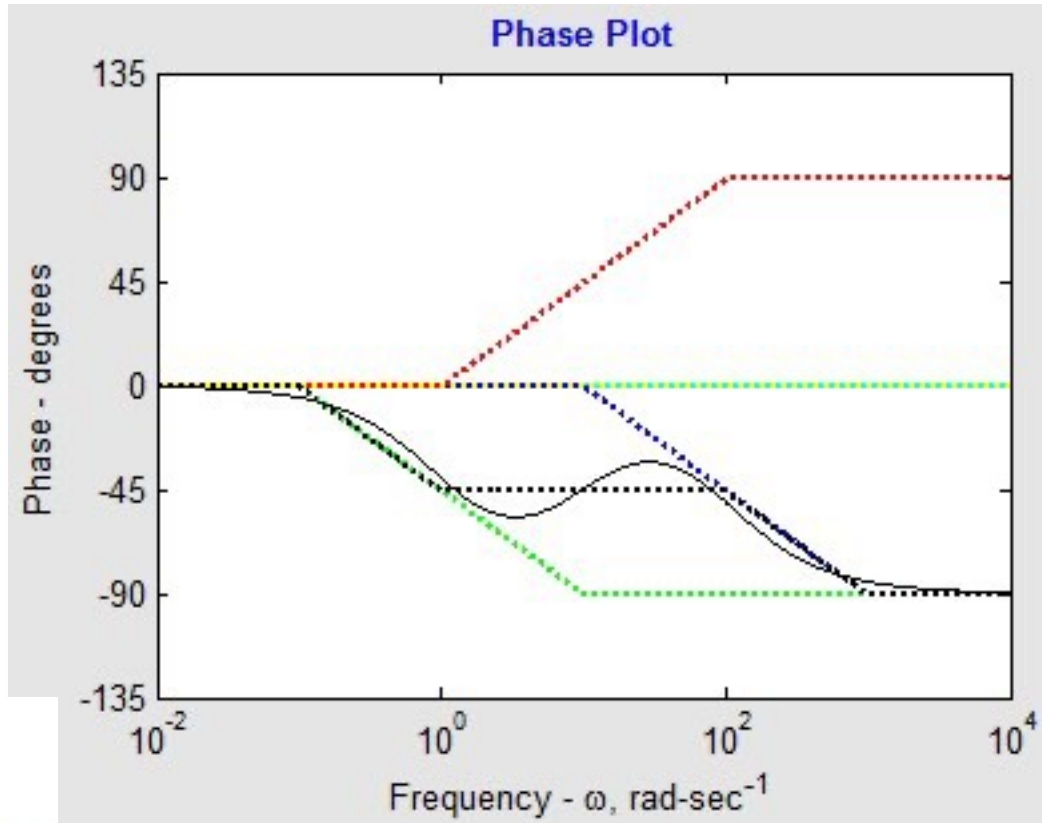



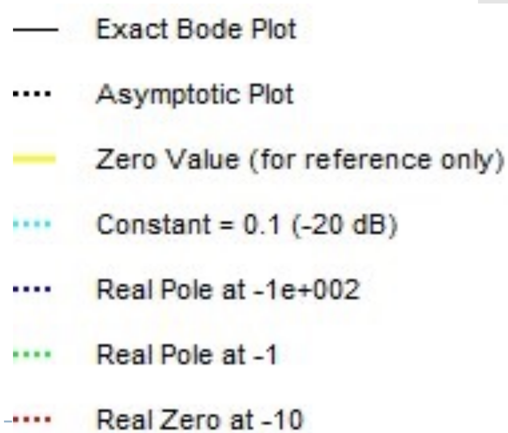
# Outros exemplos:

- $$G_2(s) = \frac{s+10}{(s+1)(s+100)}$$

$$= \frac{10\left(\frac{s}{10}+1\right)}{(s+1)\cdot 100\left(\frac{s}{100}+1\right)}$$

$$= \frac{10}{100}\cdot\frac{\left(\frac{s}{10}+1\right)}{(s+1)\left(\frac{s}{100}+1\right)}$$



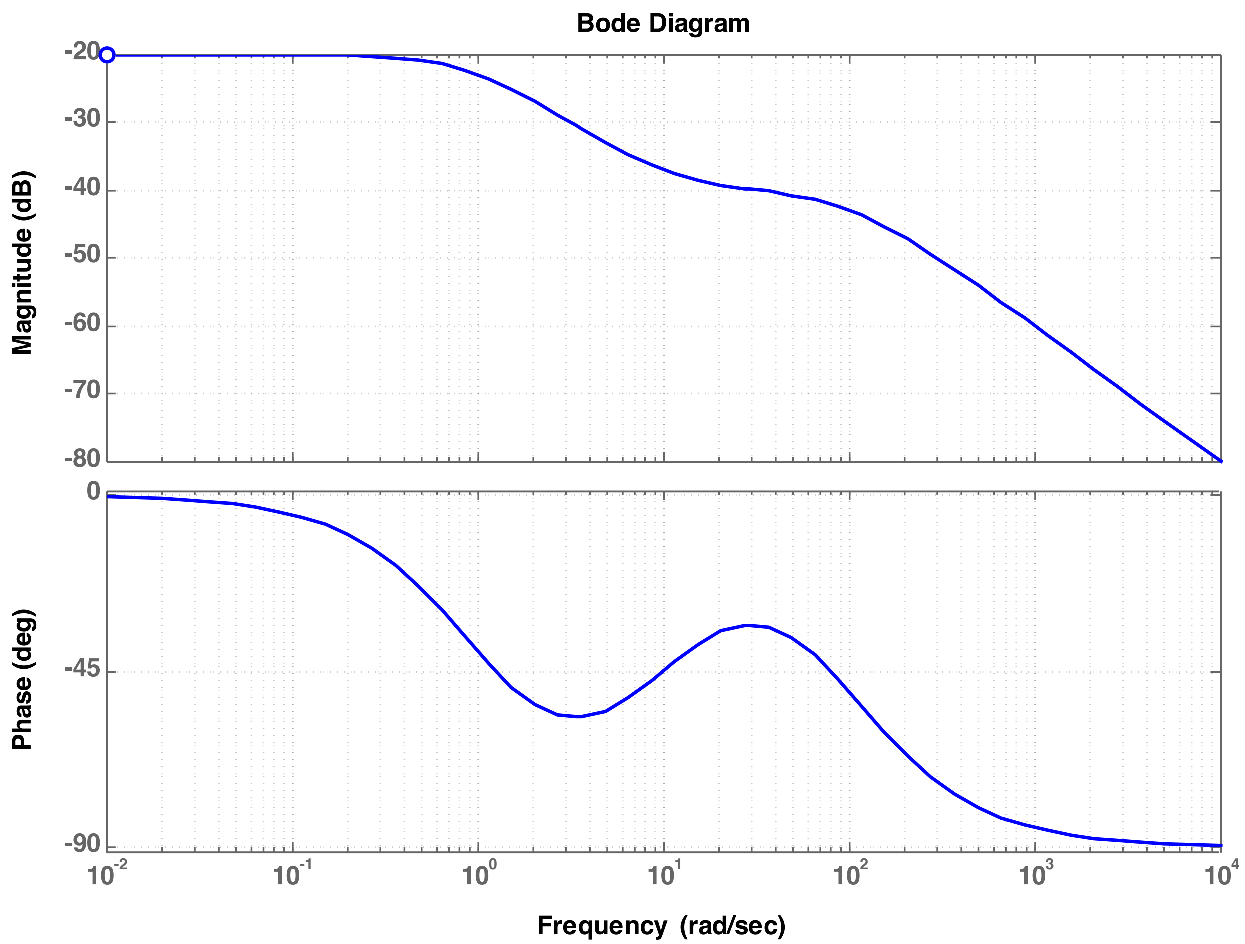



# Outros exemplos:

- $$G_2(s) = \frac{s+10}{(s+1)(s+100)}$$

$$= \frac{10\left(\frac{s}{10}+1\right)}{(s+1)\cdot 100\left(\frac{s}{100}+1\right)}$$

$$= \frac{10}{100}\cdot\frac{\left(\frac{s}{10}+1\right)}{(s+1)\left(\frac{s}{100}+1\right)}$$

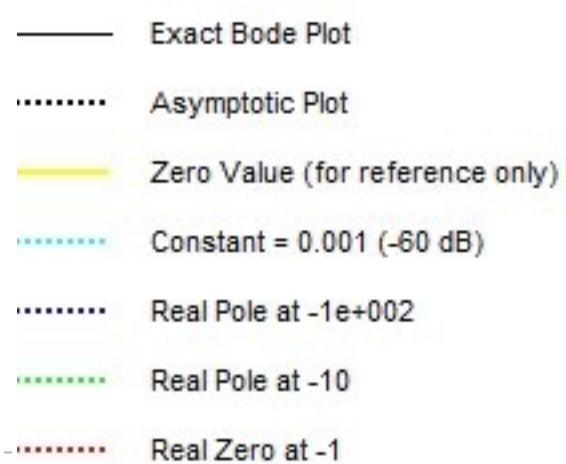


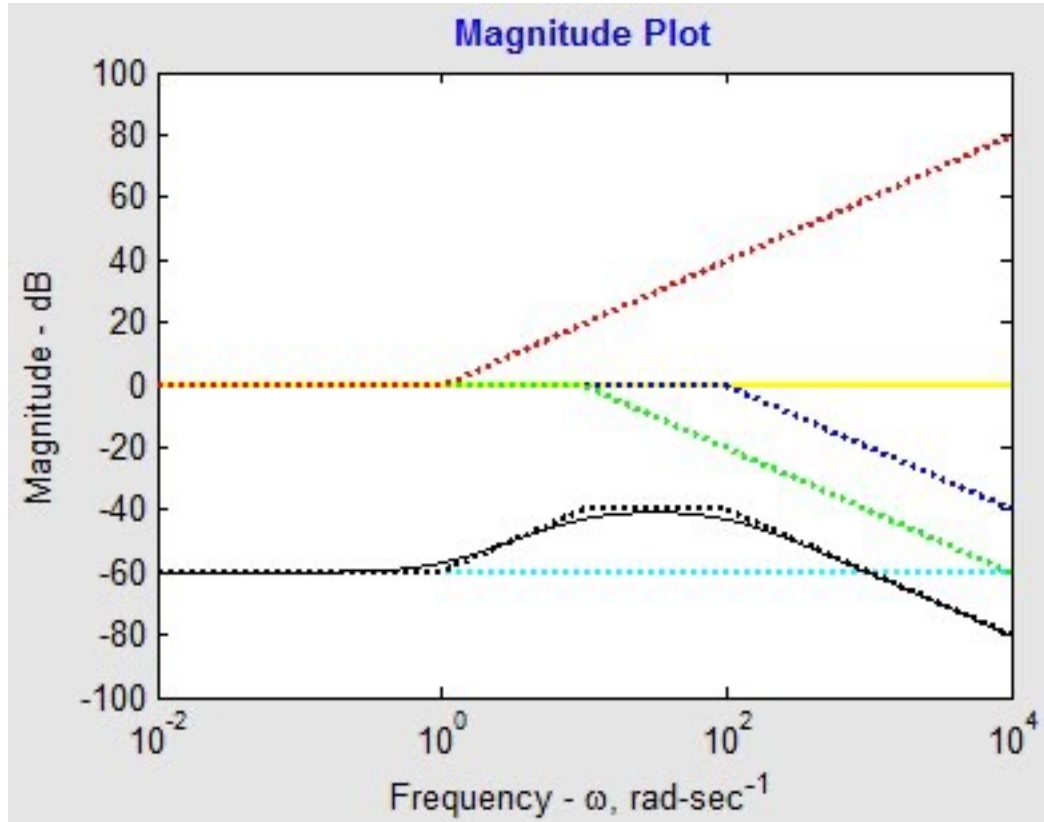
# Outros Exemplos:

$$G_3(s) = \frac{s+1}{(s+10)(s+100)}$$

$$= \frac{(s+1)}{10\left(\frac{s}{10}+1\right)\cdot 100\left(\frac{s}{100}+1\right)}$$

$$= \frac{(s+1)}{1000\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}$$



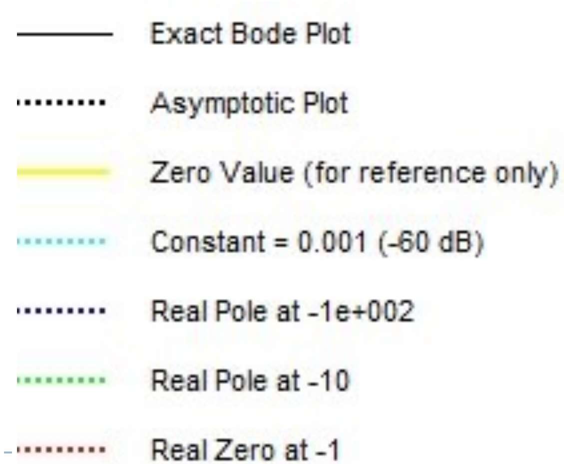



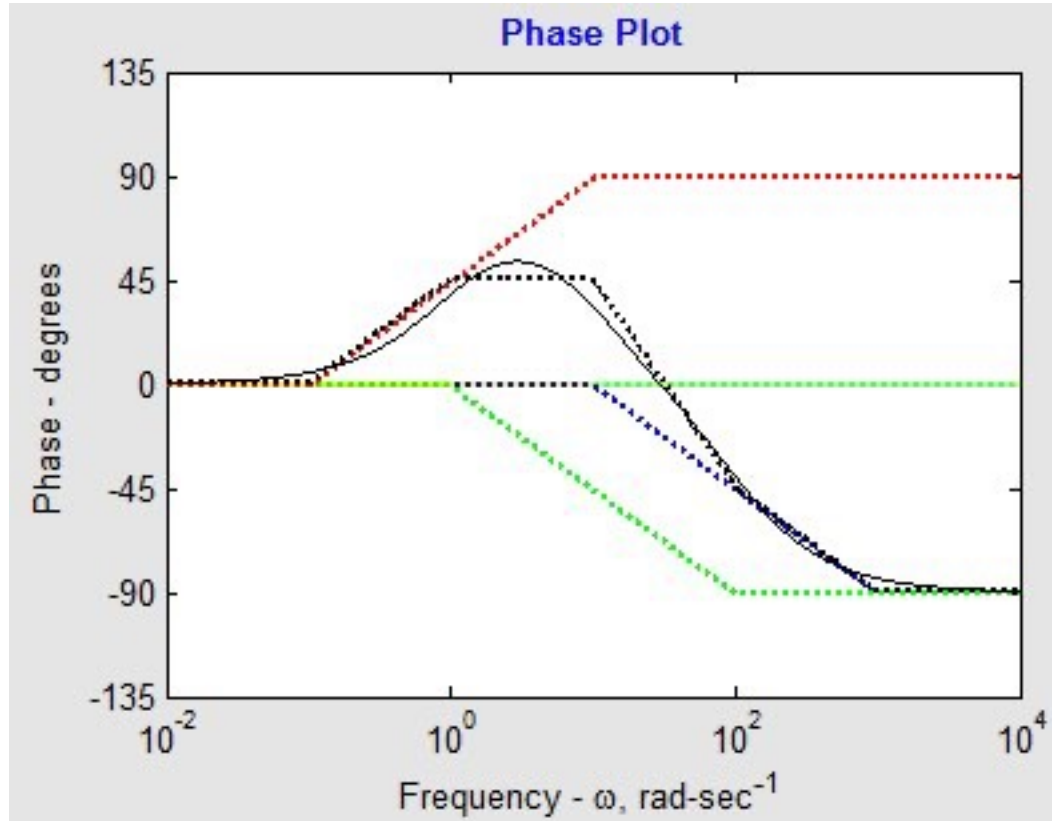
# Outros Exemplos:

$$G_3(s) = \frac{s+1}{(s+10)(s+100)}$$

$$= \frac{(s+1)}{10\left(\frac{s}{10}+1\right)\cdot 100\left(\frac{s}{100}+1\right)}$$

$$= \frac{(s+1)}{1000\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}$$



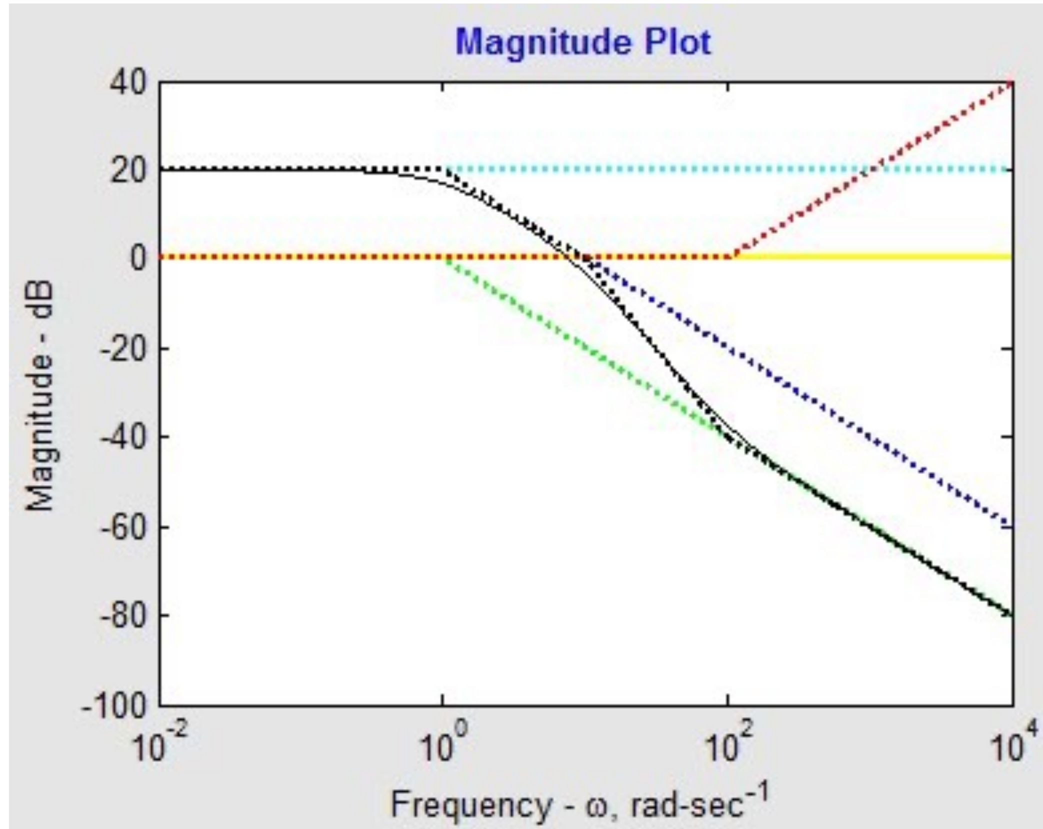



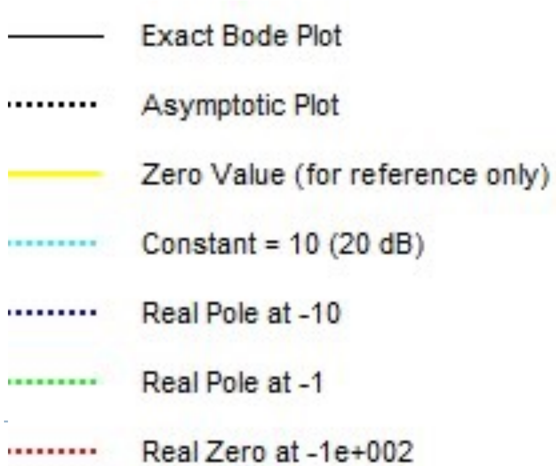
# Outros Exemplos:

$$G_4(s) = \frac{s+100}{(s+1)(s+10)}$$

$$= \frac{100\left(\dfrac{s}{100}+1\right)}{(s+1)\cdot 10\left(\dfrac{s}{10}+1\right)}$$

$$= 10\,\frac{\left(\dfrac{s}{100}+1\right)}{(s+1)\cdot\left(\dfrac{s}{10}+1\right)}$$



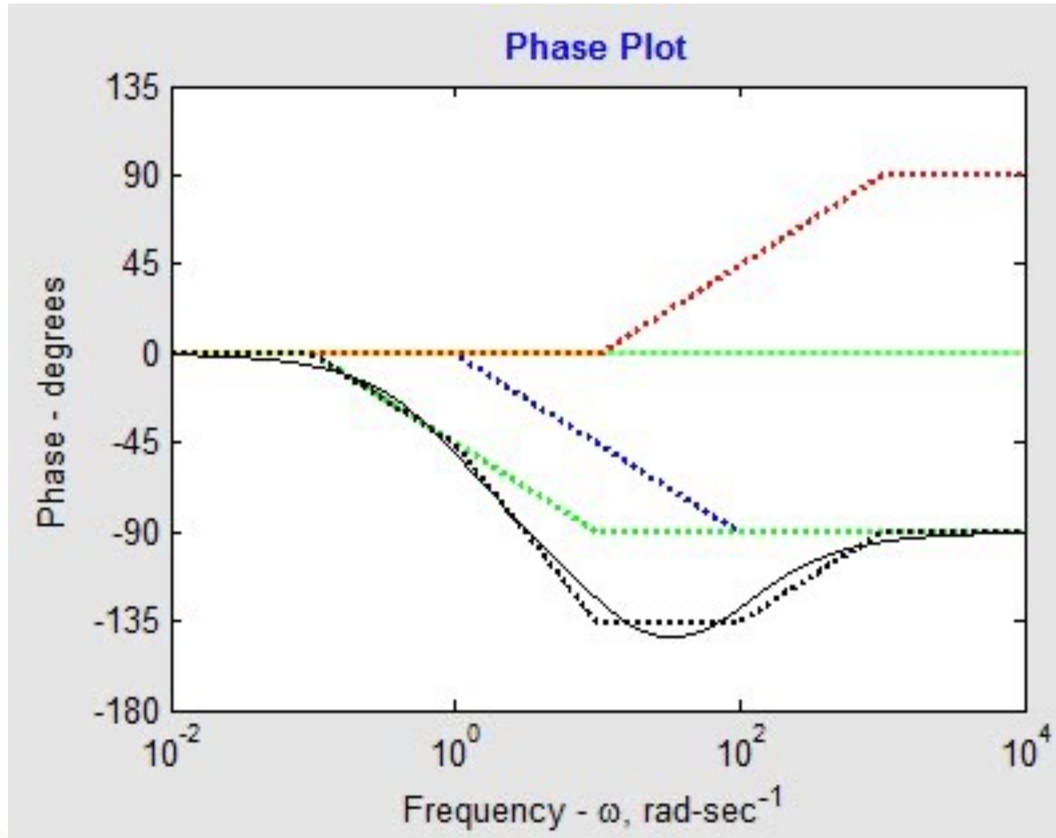



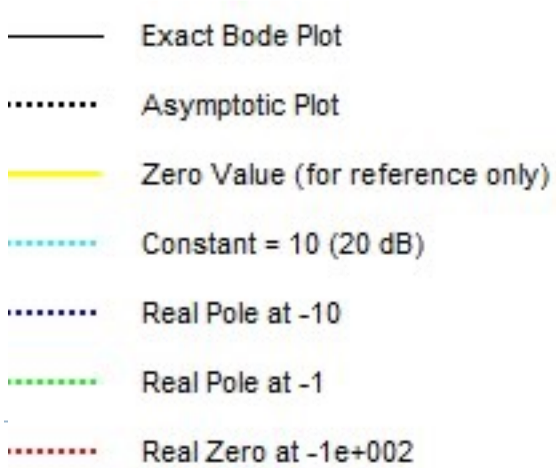
# Outros Exemplos:

$$G_4(s) = \frac{s+100}{(s+1)(s+10)}$$

$$= \frac{100\left(\frac{s}{100}+1\right)}{(s+1)\cdot 10\left(\frac{s}{10}+1\right)}$$

$$= 10\frac{\left(\frac{s}{100}+1\right)}{(s+1)\cdot\left(\frac{s}{10}+1\right)}$$



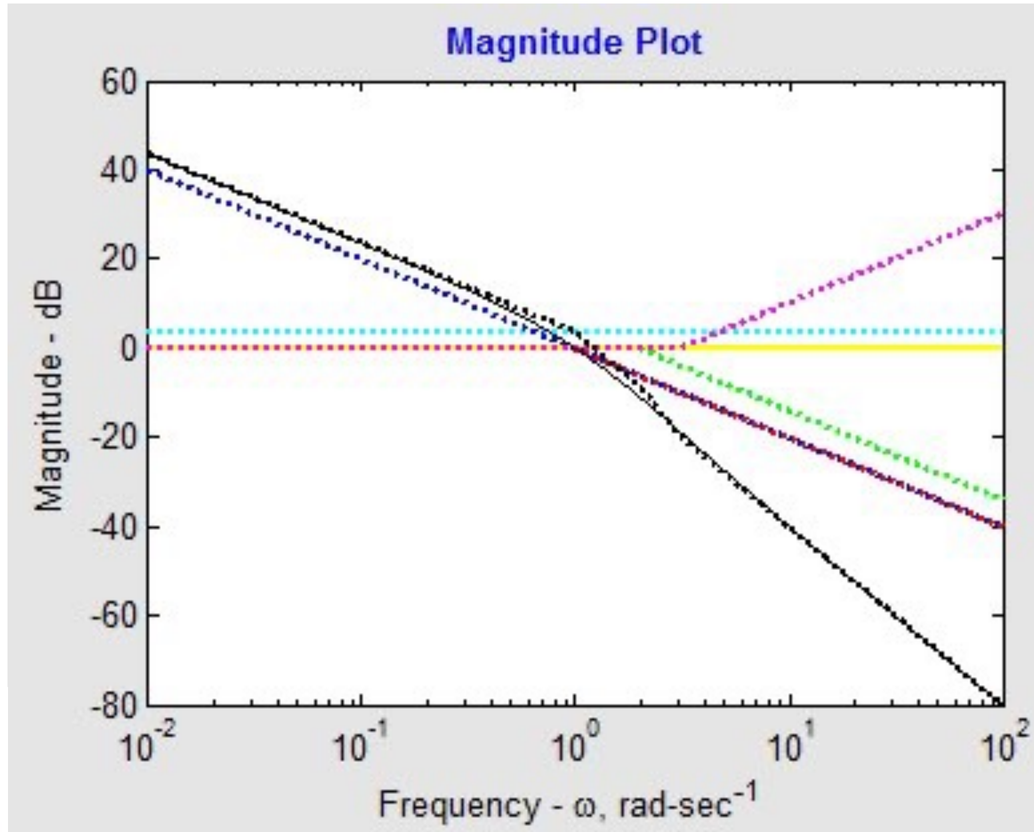



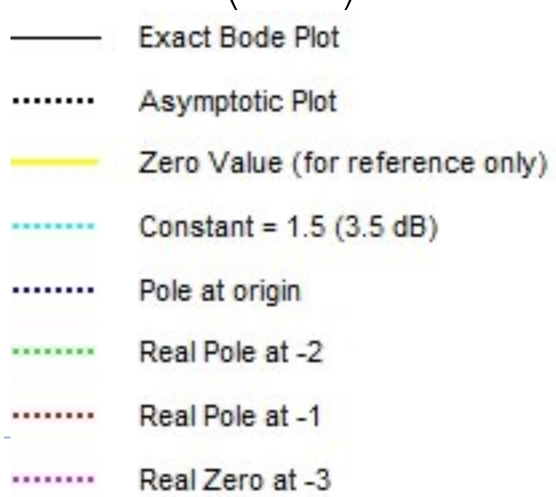
# Outros Exemplos:

$$G_5(s) = \frac{s+3}{s(s+1)(s+2)}$$

$$= \frac{3\left(\frac{s}{3}+1\right)}{s(s+1)\cdot 2\left(\frac{s}{2}+1\right)}$$

$$= \frac{3}{2}\cdot\frac{\left(\frac{s}{3}+1\right)}{s(s+1)\cdot\left(\frac{s}{2}+1\right)}$$



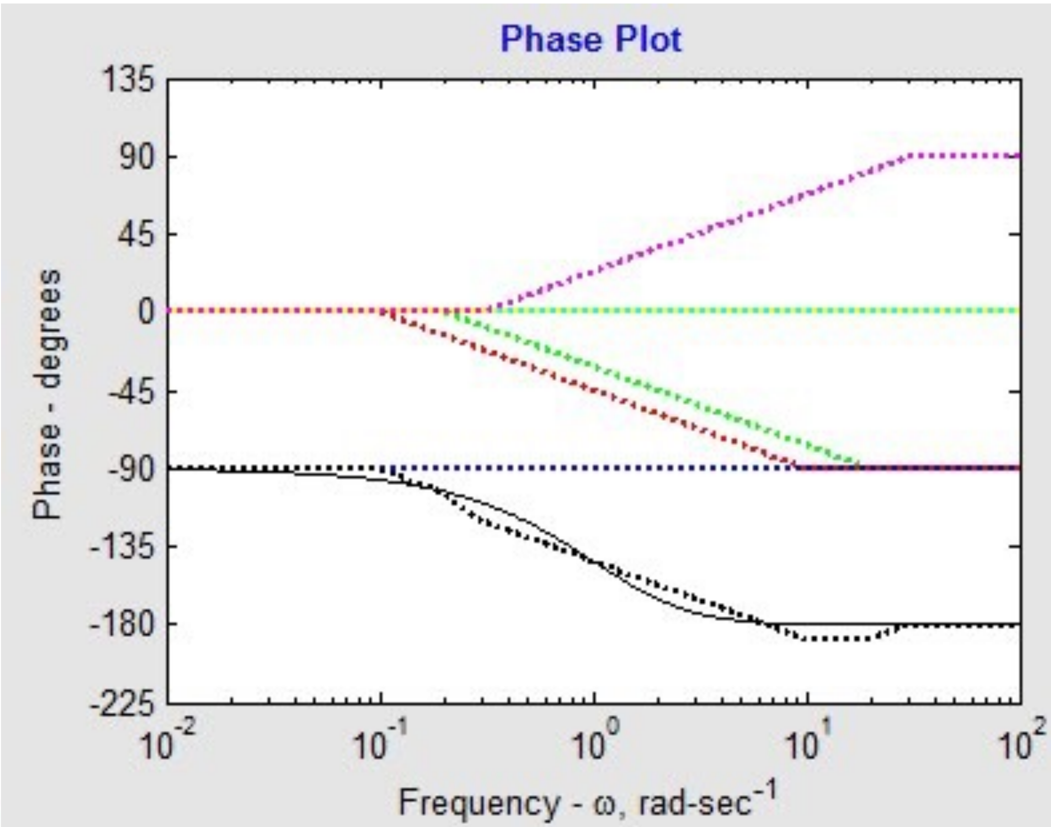

$$20\times\log(3/2) = 20\times 0{,}1761 = 3{,}5218$$

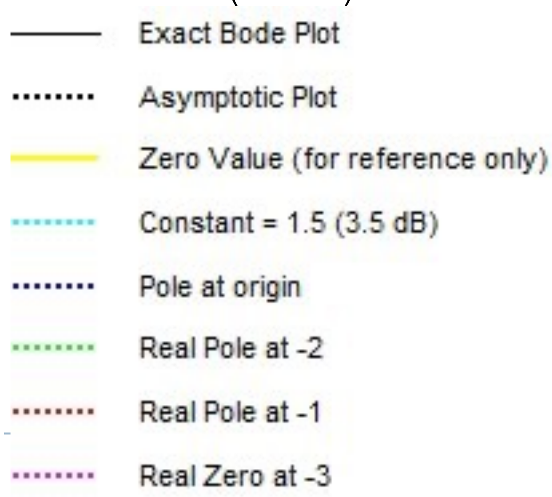
# Outros Exemplos:

- $$G_5(s) = \frac{s+3}{s(s+1)(s+2)}$$

$$= \frac{3\left(\frac{s}{3}+1\right)}{s(s+1)\cdot 2\left(\frac{s}{2}+1\right)}$$

$$= \frac{3}{2}\cdot\frac{\left(\frac{s}{3}+1\right)}{s(s+1)\cdot\left(\frac{s}{2}+1\right)}$$



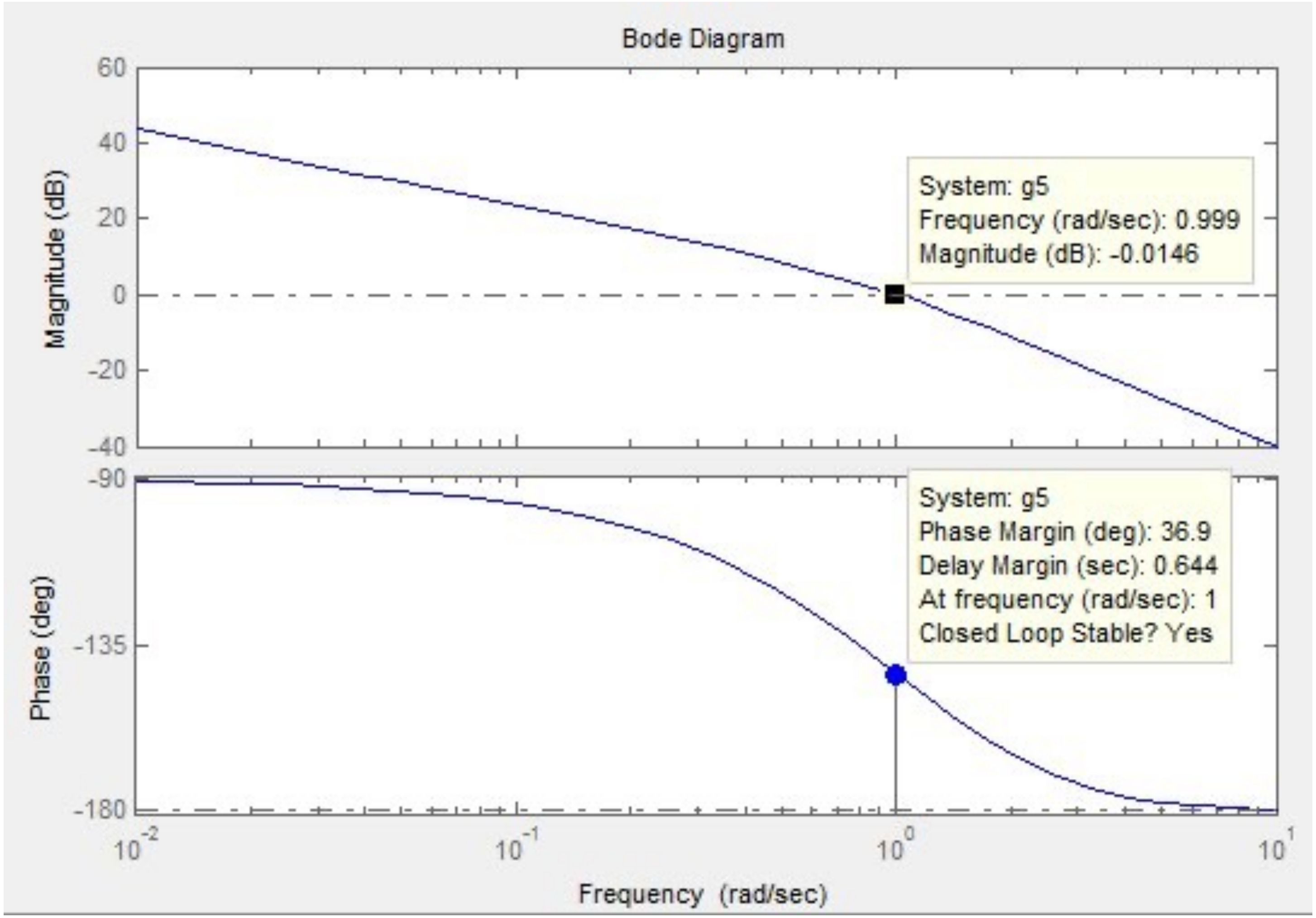



# Outros Exemplos:

$$G_5(s) = \frac{s+3}{s(s+1)(s+2)}$$

$$= \frac{3\left(\frac{s}{3}+1\right)}{s(s+1)\cdot 2\left(\frac{s}{2}+1\right)}$$

$$= \frac{3}{2}\cdot\frac{\left(\frac{s}{3}+1\right)}{s(s+1)\cdot\left(\frac{s}{2}+1\right)}$$



# Outros Exemplos:

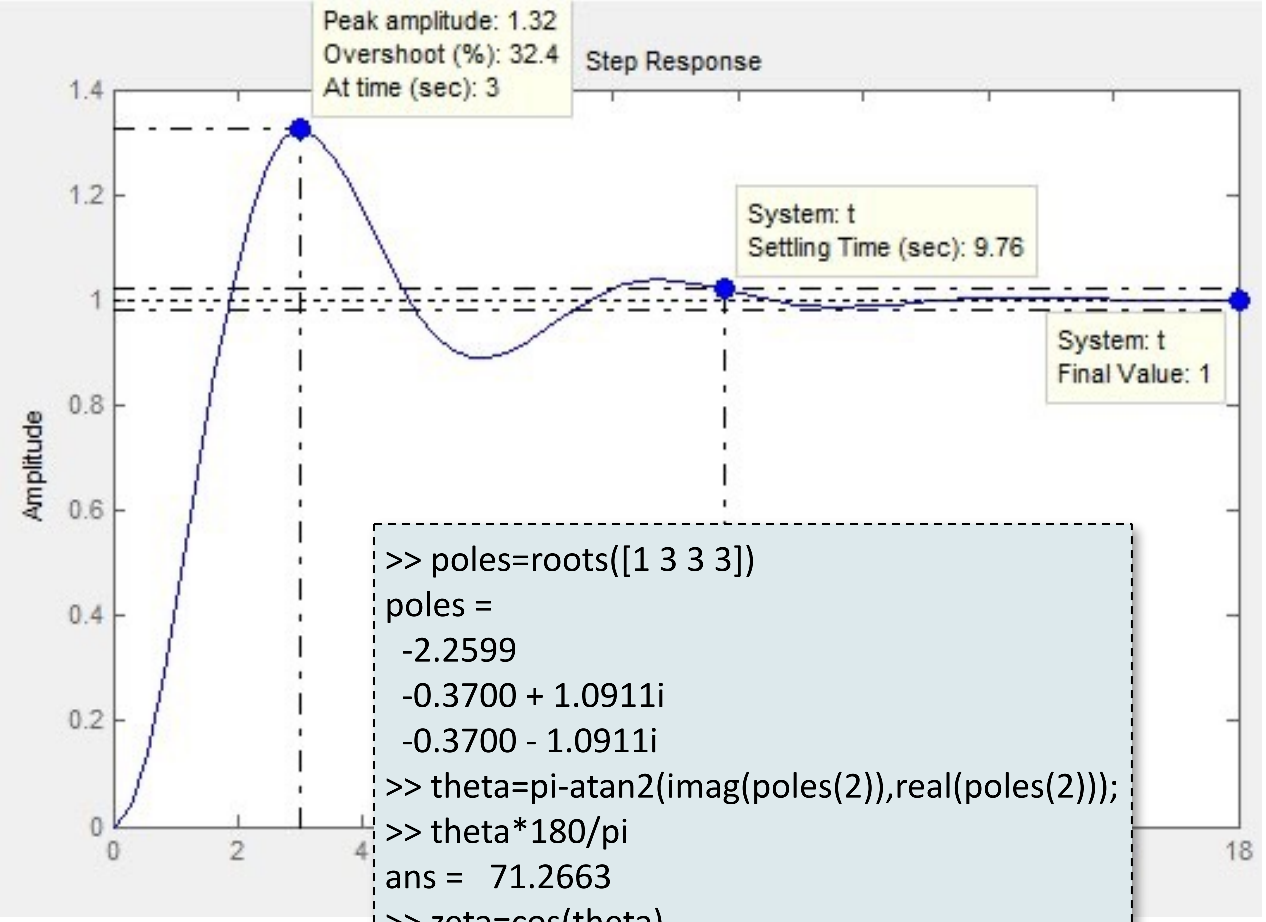
$$G_5(s) = \frac{s+3}{s(s+1)(s+2)}$$

$$T_5(s) = \frac{G(s)}{1+G(s)}$$

$$= \frac{(s+3)}{s(s+1)(s+2)+(s+3)}$$

$$= \frac{(s+3)}{s^3+3s^2+3s+2}$$

$$= \frac{(s+3)}{(s+2{,}26)(s+0{,}37 \pm j1{,}09)}$$




```
>> poles=roots([1 3 3 3])
poles =
  -2.2599
  -0.3700 + 1.0911i
  -0.3700 - 1.0911i
>> theta=pi-atan2(imag(poles(2)),real(poles(2)));
>> theta*180/pi
ans =   71.2663
>> zeta=cos(theta)
zeta =    0.3212
>> OS=exp(-(zeta*pi)/(sqrt(1-zeta*zeta)))*100
OS =   34.4580
>>
```

# Sistemas com polos complexos

- Seja:

$$G(s) = s^2 + 2\zeta w_n s + w_n^2 = w_n^2\left(\frac{s^2}{w_n^2} + 2\zeta\,\frac{s}{w_n} + 1\right)$$

- Em baixas frequências:

$$G(s) \approx w_n^2 = w_n^2\angle 0^o \qquad 20\log M = 20\log\left|G(jw)\right| = 20\log w_n^2$$

- Em altas frequências:

$$20\log M = 20\log\left|G(jw)\right| = 20\log w^2 = 40\log w$$

$$G(s) \approx s^2$$

$$G(jw) \approx -w^2 = w^2\angle 180^o$$

- Detalhes:
  - $w_n$: frequência de corte (quebra).
  - Fase em $w_n$:

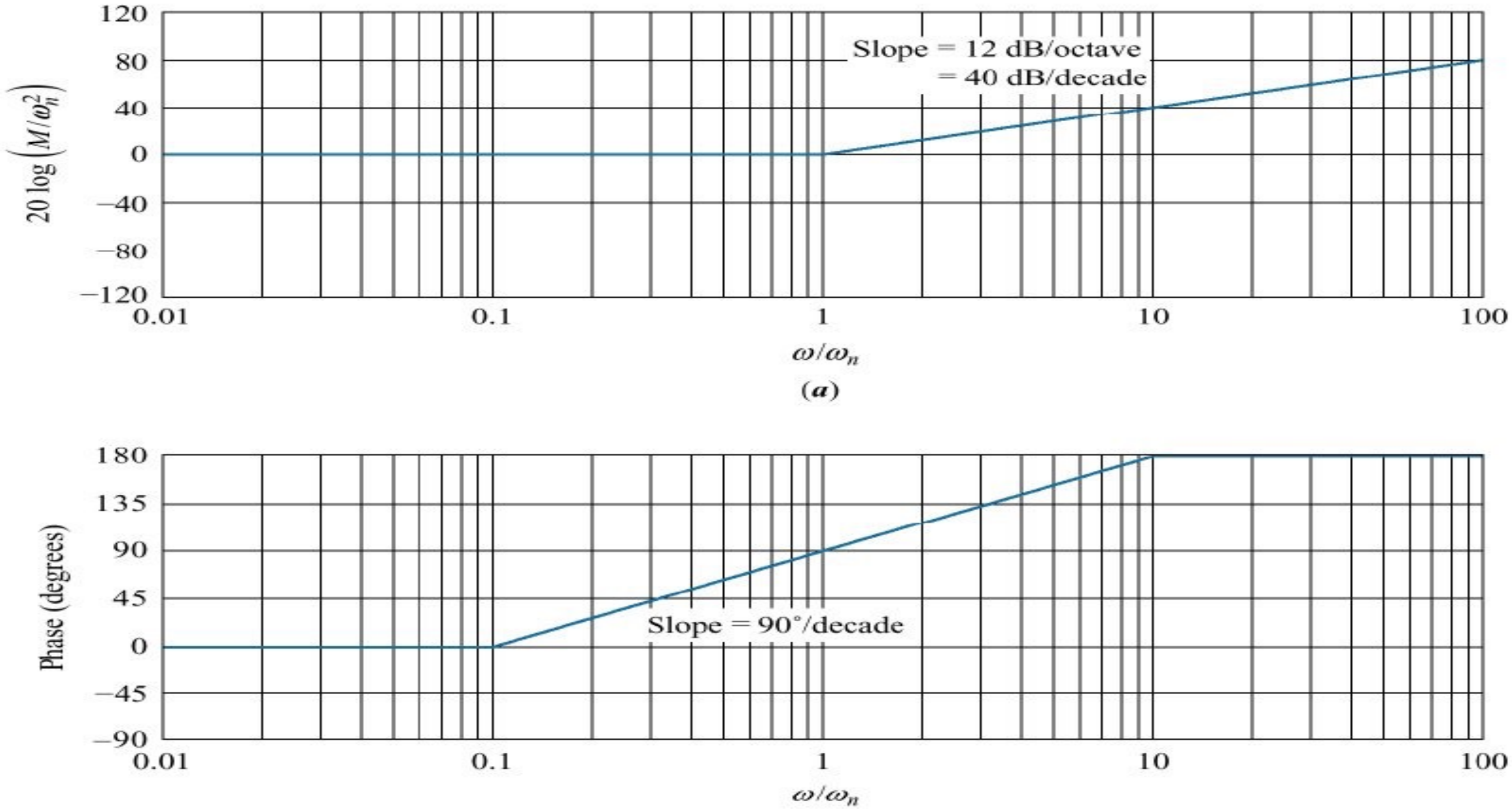
$$G(jw) = s^2 + 2\zeta w_n s + w_n^2\Big|_{s\to jw} = \left(w_n^2 - w^2\right) + j2\zeta w_n w$$

em wn o resultado é: $j2\zeta w_n^2$ assim a fase na frequência natural é de +90º

# Sistemas com polos complexos

- Seja: $$G(s) = s^2 + 2\zeta w_n s + w_n^2 = w_n^2\left(\frac{s^2}{w_n^2} + 2\zeta\frac{s}{w_n} + 1\right)$$
- Em baixas frequências: $G(s) \approx w_n^2 = w_n^2\angle 0^o$ $\quad 20\log M = 20\log|G(jw)| = 20\log w_n^2$
- Em altas frequências: $G(s) \approx s^2$ $\quad 20\log M = 20\log|G(jw)| = 20\log w^2 = 40\log w$

$G(jw) \approx -w^2 = w^2\angle 180^o$



# Sistemas com polos complexos
# Correções em diagrama assintótico..

- Seja:

$$G(s) = s^2 + 2\zeta w_n s + w_n^2 = w_n^2 \left( \frac{s^2}{w_n^2} + 2\zeta \frac{s}{w_n} + 1 \right)$$

- Um polinômio de 1ª-ordem resulta numa diferença não superior a 3,01 dB na magnitude e 5,71° em relação à fase (no ponto do polo).

- Um polinômio de 2ª-ordem pode implicar maior disparidade, depende do valor de $\zeta$ (na localização dos polos complexos):

$$G(jw) = s^2 + 2\zeta w_n s + w_n^2 \Big|_{s \to jw} = \left( w_n^2 - w^2 \right) + j2\zeta w_n w$$

$$M = \sqrt{(w_n^2 - w^2)^2 + (2\zeta w_n w)^2}$$

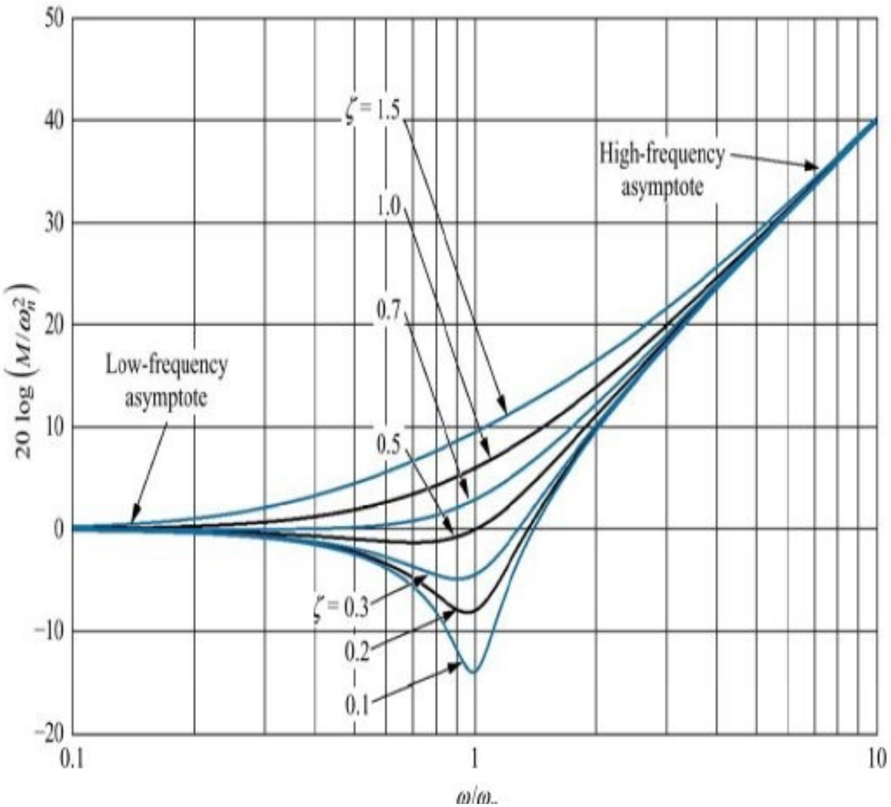
$$Fase = \tan^{-1}\left( \frac{2\zeta w_n w}{w_n^2 - w^2} \right)$$

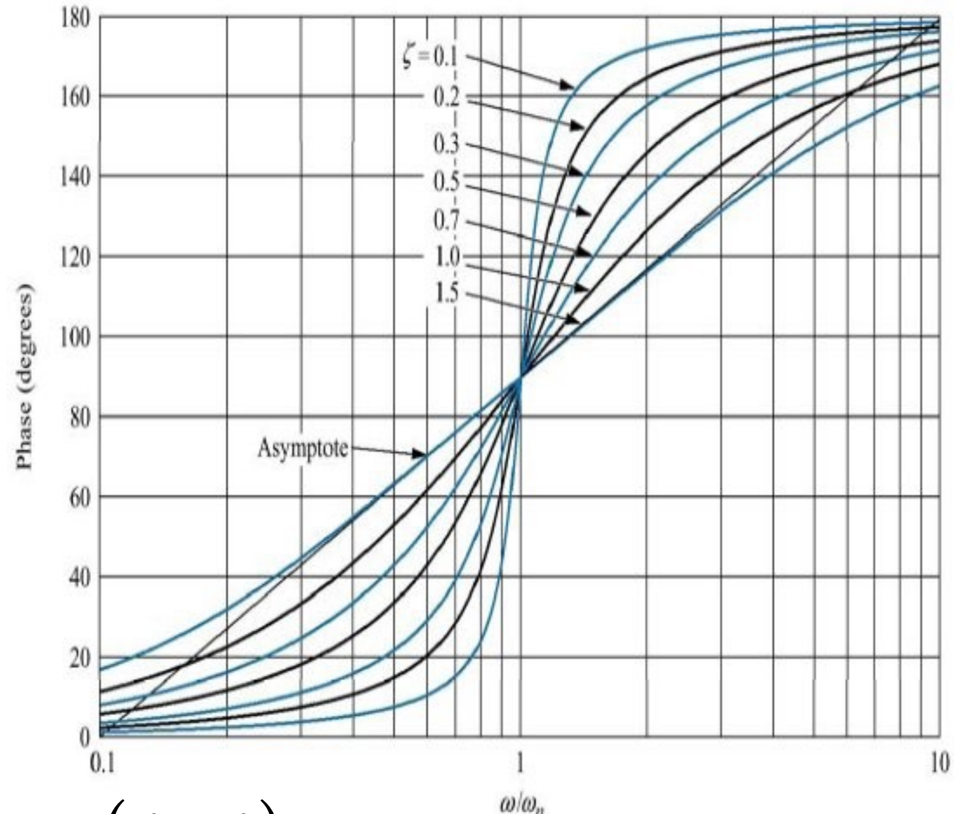
# Sistemas com polos complexos
# Correções em diagrama assintótico..

- **Seja:** $G(s) = s^2 + 2\zeta w_n s + w_n^2 = w_n^2\left(\frac{s^2}{w_n^2} + 2\zeta\frac{s}{w_n} + 1\right)$

$$M = \sqrt{(w_n^2 - w^2)^2 + (2\zeta w_n w)^2}$$

$$Fase = \tan^{-1}\left(\frac{2\zeta w_n w}{w_n^2 - w^2}\right)$$



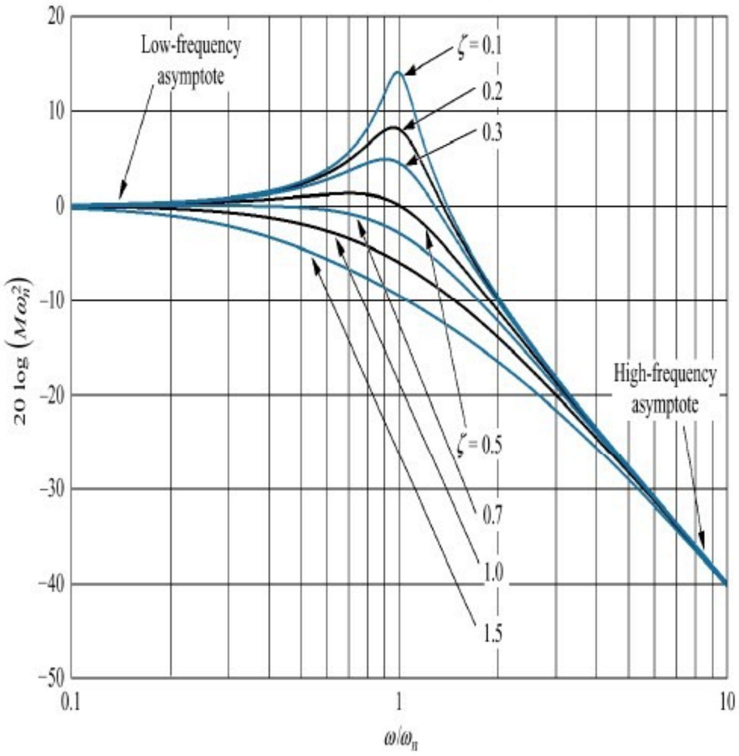



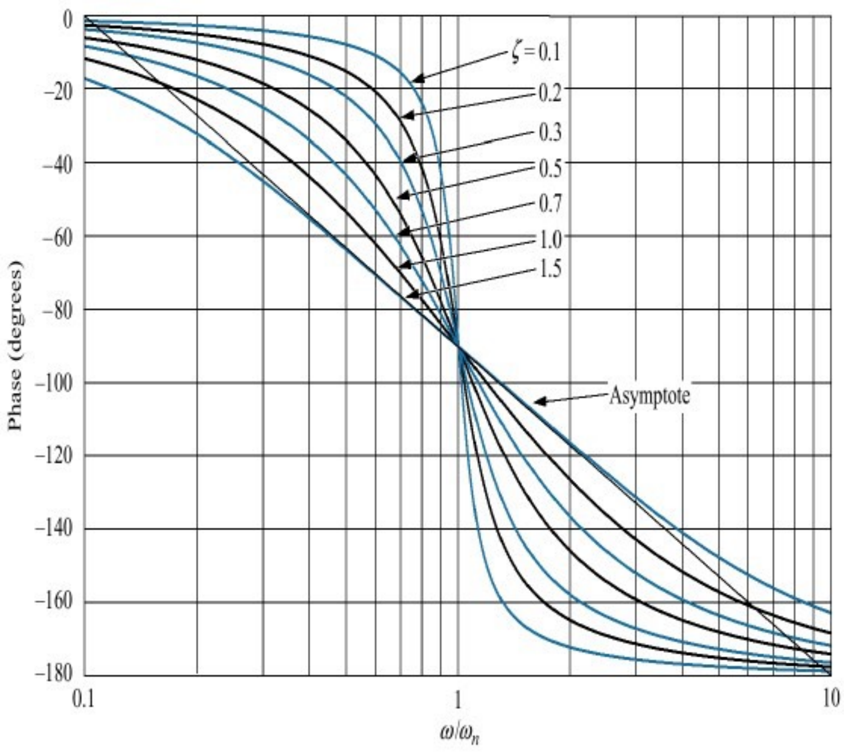

$$G(jw) = s^2 + 2\zeta w_n s + w_n^2\Big|_{s \to jw} = \left(w_n^2 - w^2\right) + j2\zeta w_n w$$

# Sistemas com polos complexos
# Correções em diagrama assintótico..

$$G(s) = 1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$$

- The slope is –40dB/decade.
- The normalized magnitude at the scaled natural frequency is $-20\log 2\zeta\omega_n^2$



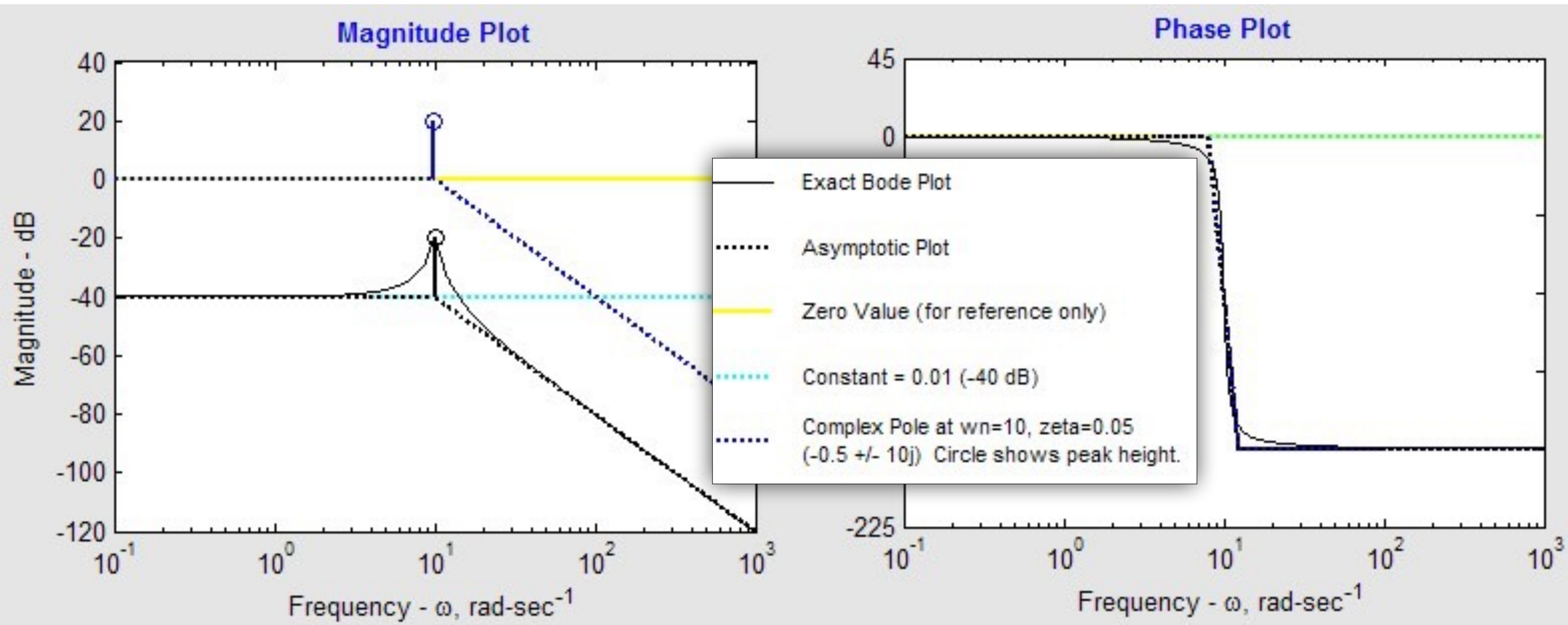
# Exemplo polos complexos

- $G(s) = \dfrac{1}{s^2 + s + 100}$

  $G(s) = \dfrac{1}{(s + 0{,}5 + j10)(s + 0{,}5 - j10)}$

$$G(s) = s^2 + 2\zeta w_n s + w_n^2 = w_n^2\left(\frac{s^2}{w_n^2} + 2\zeta\,\frac{s}{w_n} + 1\right)$$

$$w_n = \sqrt{100} = 10 \qquad \zeta = 1/2w_n = 1/20 = 0{,}05$$




20*log10(1/100)=-40 db

```
>> roots([1 3 50])
ans =
  -1.5000 + 6.9101i
  -1.5000 - 6.9101i
>>
```

# Problemas sugeridos:

$$H_1(s) = 30\frac{s+10}{s^2+3s+50}$$

$$H(s) = 30\frac{s+10}{s^2+3s+50} = 30\frac{10}{50}\frac{\frac{s}{10}+1}{\frac{s^2}{50}+\frac{3}{50}s+1} = 6\frac{\frac{s}{10}+1}{\frac{s^2}{50}+\frac{3}{50}s+1}$$

- valor constante = 6,
- um zero e s=-10,
- e par de polos complexos conjugados em: raízes de: $s^2+3s+50=0$;
- Polos complexos em s=-1,5 ± *j*6,9 (onde *j*=sqrt(-1)).

Uma maneira mais comum (e útil para nossos propósitos) de expressar isso é usar a notação padrão para um polinômio de segunda ordem:

$$\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right)+1 \qquad \omega_0 = \sqrt{50} = 7.07, \quad \zeta = \frac{3\sqrt{50}}{2\cdot 50} = 0.21$$

Exact Bode Plot

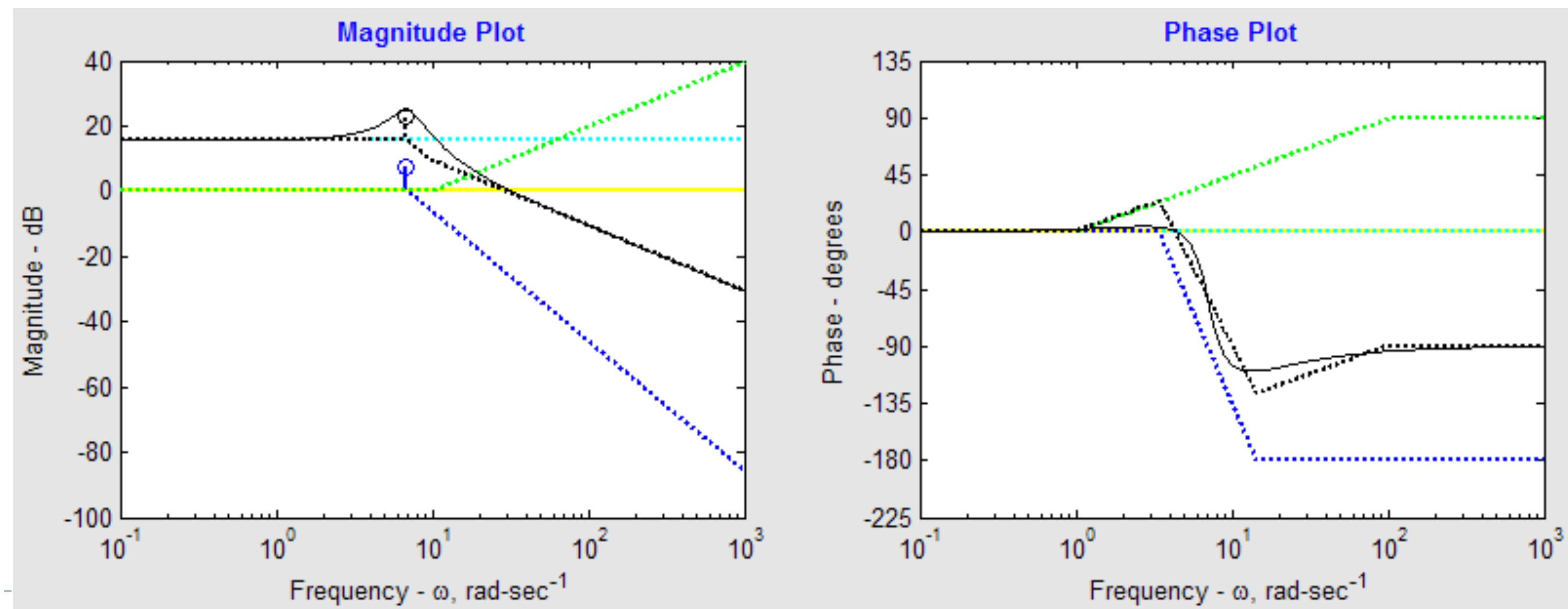
Asymptotic Plot

Zero Value (for reference only)

Constant = 6 (16 dB)

Complex Pole at wn=7.1, zeta=0.21
(-1.5 +/- 6.9j) Circle shows peak height.

Real Zero at -10



# Problemas sugeridos:

$$H_2(s) = 30\frac{5s}{s^2+3s+50}$$

Exact Bode Plot

Asymptotic Plot

Zero Value (for reference only)

Constant = 3 (9.5 dB)

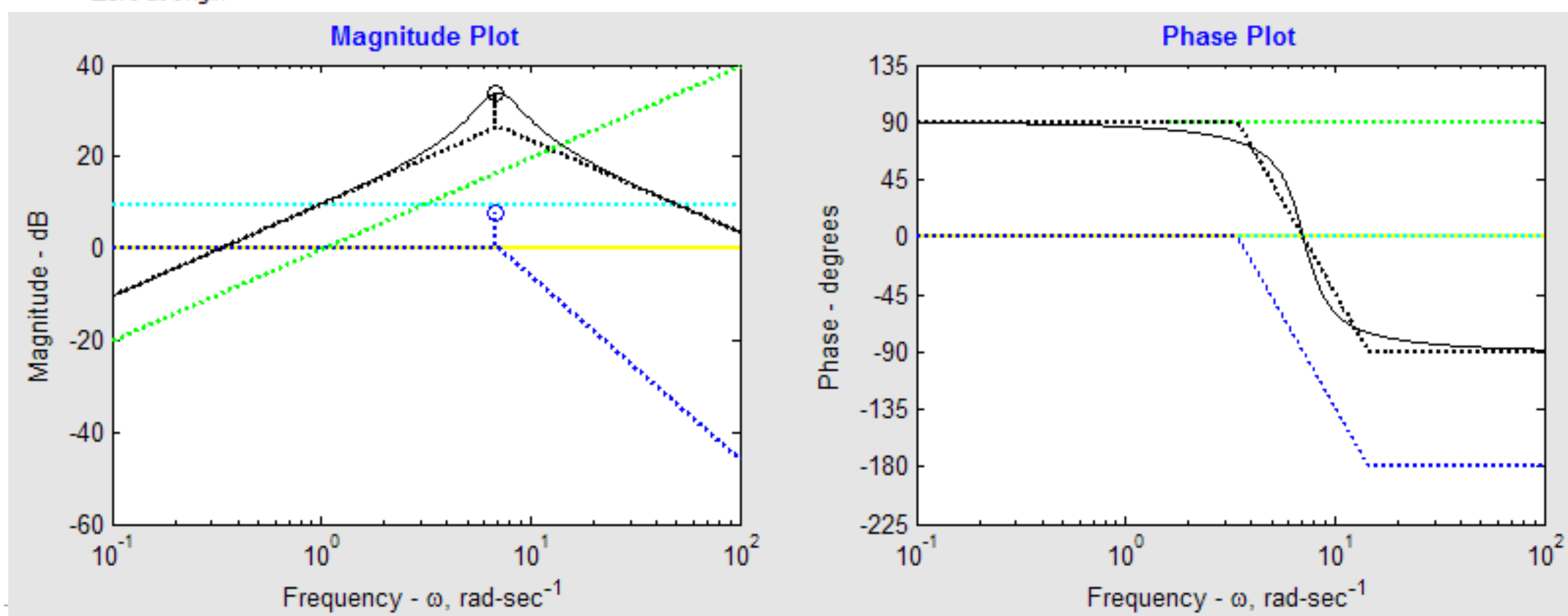
Complex Pole at wn=7.1, zeta=0.21
(-1.5 +/- 6.9j) Circle shows peak height.

Zero at origin

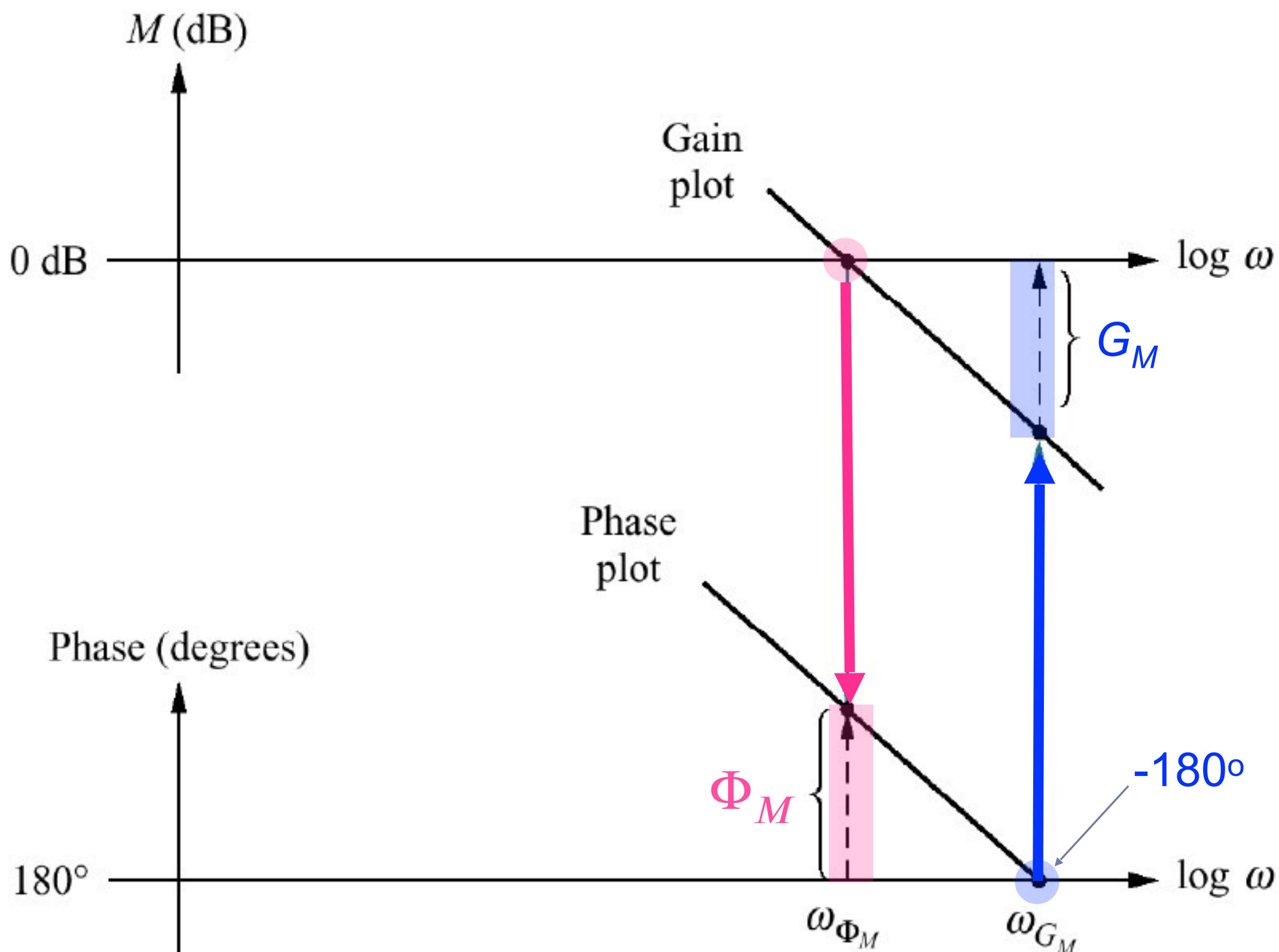
$$\mathrm{H}(s) = 30\frac{s+10}{s^2+3s+50} = 30\frac{10}{50}\frac{\frac{s}{10}+1}{\frac{s^2}{50}+\frac{3}{50}s+1} = 6\frac{\frac{s}{10}+1}{\frac{s^2}{50}+\frac{3}{50}s+1}$$

- valor constante = 6,
- um zero e s=-10,
- e par de polos complexos conjugados em: raízes de: $s^2+3s+50=0$, ou:

$$\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1 \qquad \omega_0 = \sqrt{50} = 7.07, \quad \zeta = \frac{3\sqrt{50}}{2\cdot 50} = 0.21$$

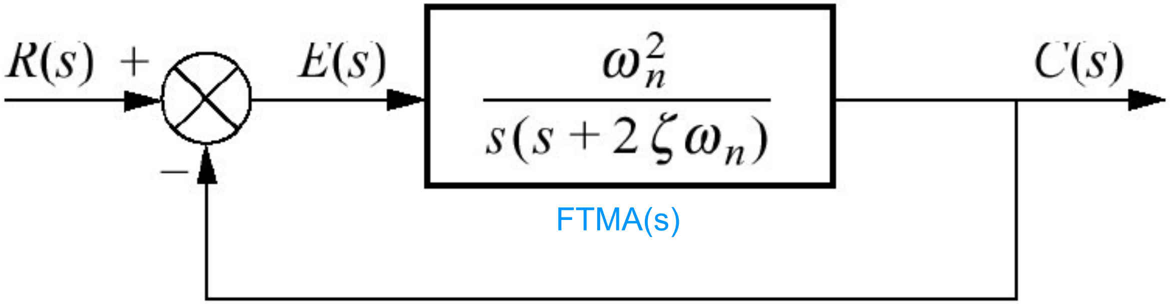


# Estabilidade, **Margem de Ganho**, $G_M$ e **Margem de Fase**, $\Phi_M$ através do Diagrama de Bode…



# Relação entre Transitório de Malha Fechada e Resposta em Frequência de malha aberta

- Fator de amortecimento, ζ e resposta em frequência de malha fechada, *T*(*s*):




$$\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$ ← FTMF(s)

$$M = |T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

$$M_P = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\omega_P = \omega_n\sqrt{1-2\zeta^2}$$

# Relação entre Transitório de Malha Fechada e Resposta em Frequência de malha aberta

- Através do Diagrama de Bode de um sistema ainda em malha aberta, $G(s)$, se pode prever o porcentual de sobressinal, $\%OS$, do sistema em malha fechada, $T(s)$:
  - Este valor se pode obter a partir da margem de fase do sistema em malha aberta:

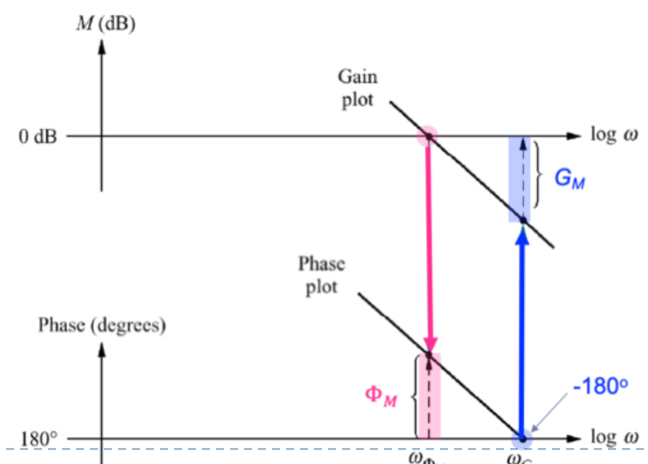
$$G(s) \xrightarrow{\text{Bode}} \Phi_M$$

$$\Phi_M \rightarrow \zeta$$

$$T(s) \leftarrow \zeta \rightarrow \%OS = e^{-\left(\zeta\pi / \sqrt{1-\zeta^2}\right)} \times 100$$

Se é possível variar o $\Phi_M$, então se pode variar o %OS!

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$



# Relação entre Transitório de Malha Fechada e Resposta em Frequência de malha aberta

Sistema malha aberta:

$$G(s) = \frac{w_n^2}{s(s+2\zeta w_n)}$$

Sistema malha fechada:

$$T(s) = \frac{w_n^2}{s^2 + 2\zeta w_n + w_n^2}$$

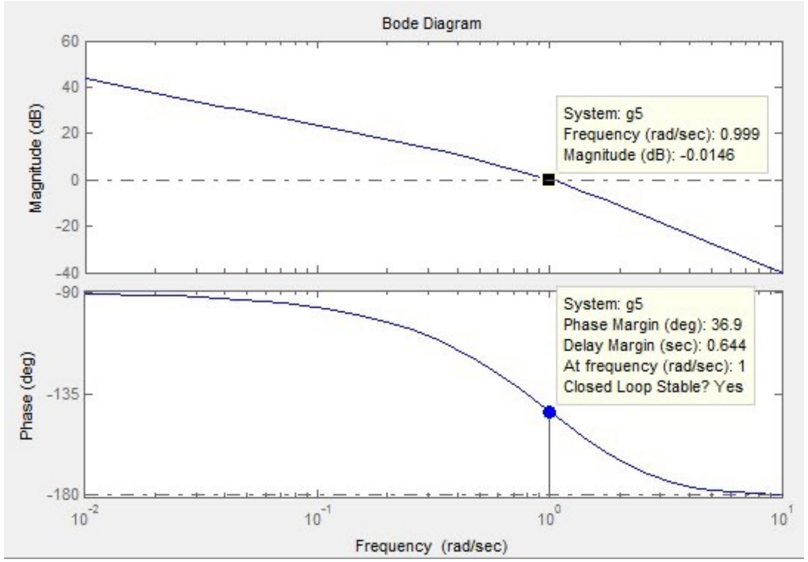
Encontrando frequência $w_1$ onde $|G(jw)| = 1$

$$|G(jw)| = \frac{w_n^2}{\left|-w^2 + j2\zeta w_n w\right|} = 1$$

$$w_1 = w_n\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}$$

$$\angle G(jw) = -90 - \tan^{-1}\left(\frac{w_1}{2\zeta w_n}\right)$$

$$= -90 - \tan^{-1}\left(\frac{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}{2\zeta}\right)$$




Como $\Phi M = \angle G(jw) - 180^o$ :

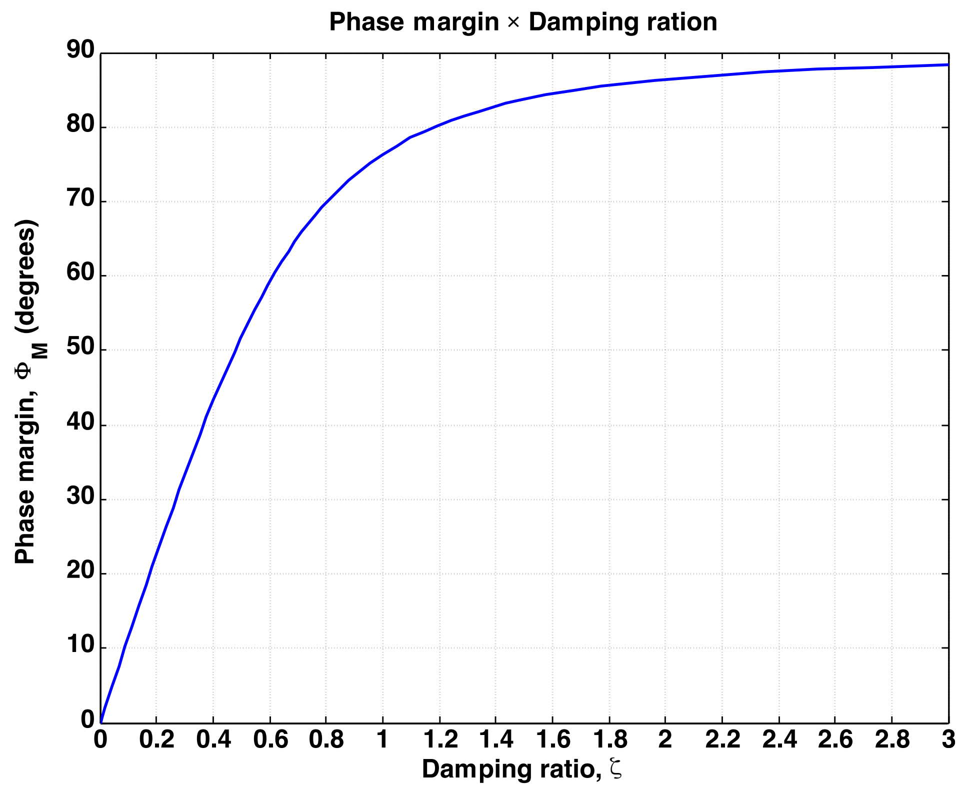
$$\Phi_M = 90 - \tan^{-1}\left(\frac{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}{2\zeta}\right)$$

$$\Phi_M = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}\right)$$

# Relação entre Transitório de Malha Fechada e Resposta em Frequência de malha aberta

$$\Phi_M = 90 - \tan^{-1}\left(\frac{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}{2\zeta}\right)$$

$$\Phi_M = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}\right)$$

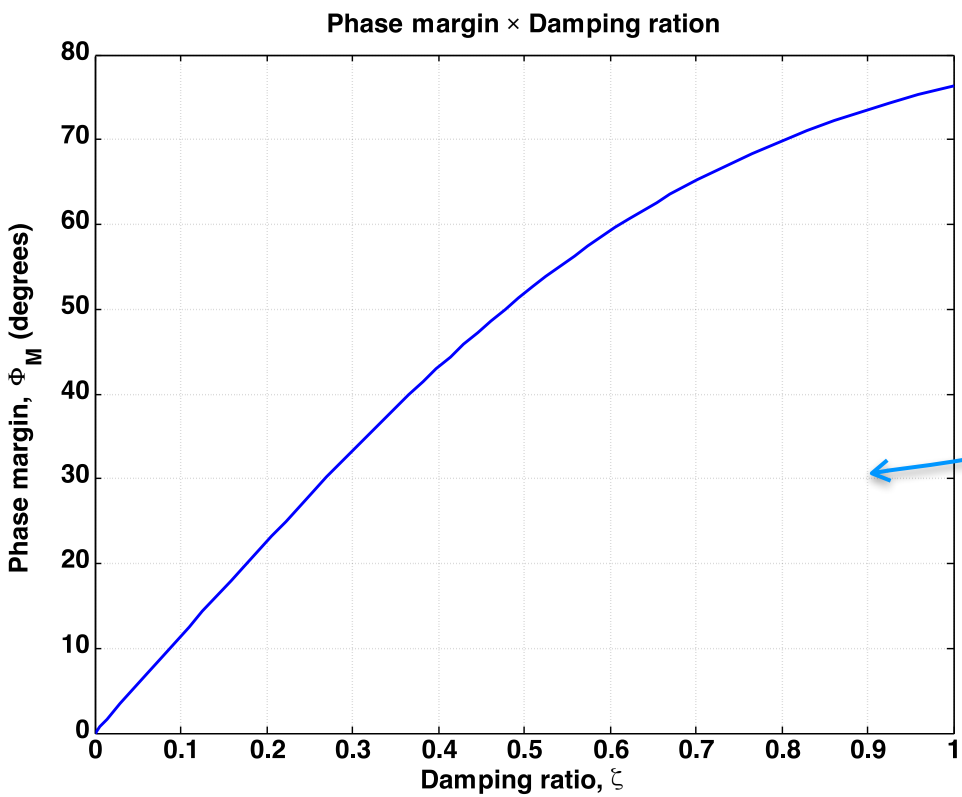


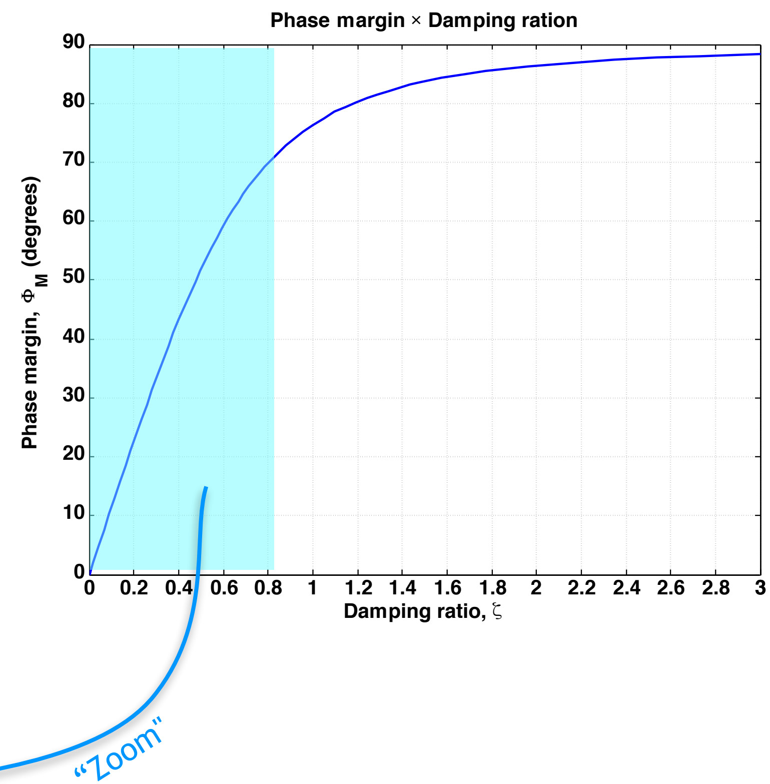

```
>> fplot(@(zeta) atan2(2*zeta,sqrt(-2*zeta*zeta+sqrt(1+4*zeta^4)))*180/pi, [0 3] )
>> grid
>> title('Phase margin \times Damping ration')
>> xlabel('Damping ratio, \zeta')
>> ylabel('Phase margin, \Phi_M (degrees)')
```

# Relação entre Transitório de Malha Fechada e Resposta em Frequência de malha aberta

$$\Phi_M = 90 - \tan^{-1}\left(\frac{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}{2\zeta}\right)$$

$$\Phi_M = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}\right)$$



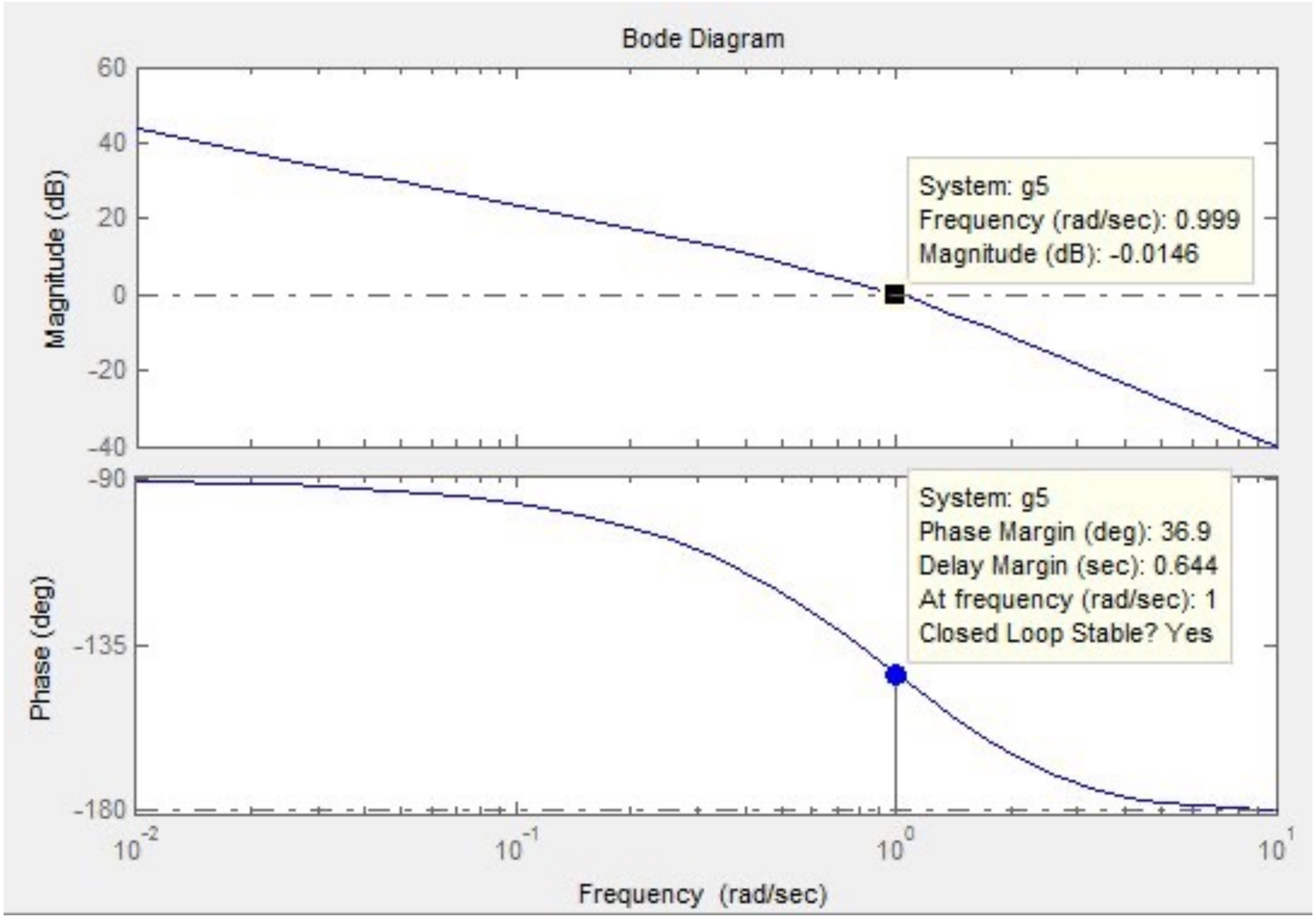



# Outros Exemplos:

$$G_5(s) = \frac{s+3}{s(s+1)(s+2)}$$

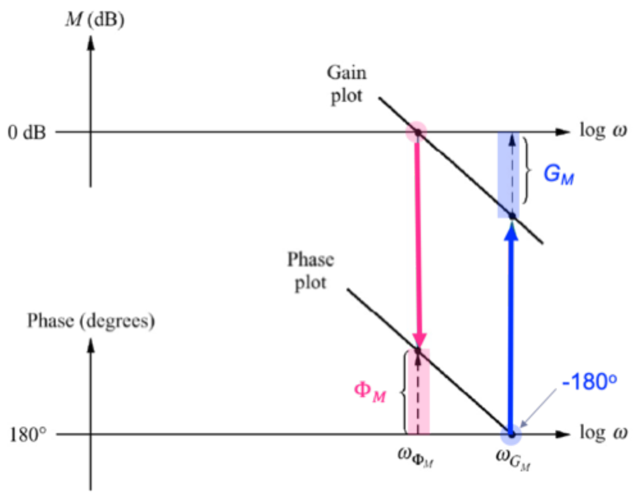
$$= \frac{3\left(\frac{s}{3}+1\right)}{s(s+1)\cdot 2\left(\frac{s}{2}+1\right)}$$

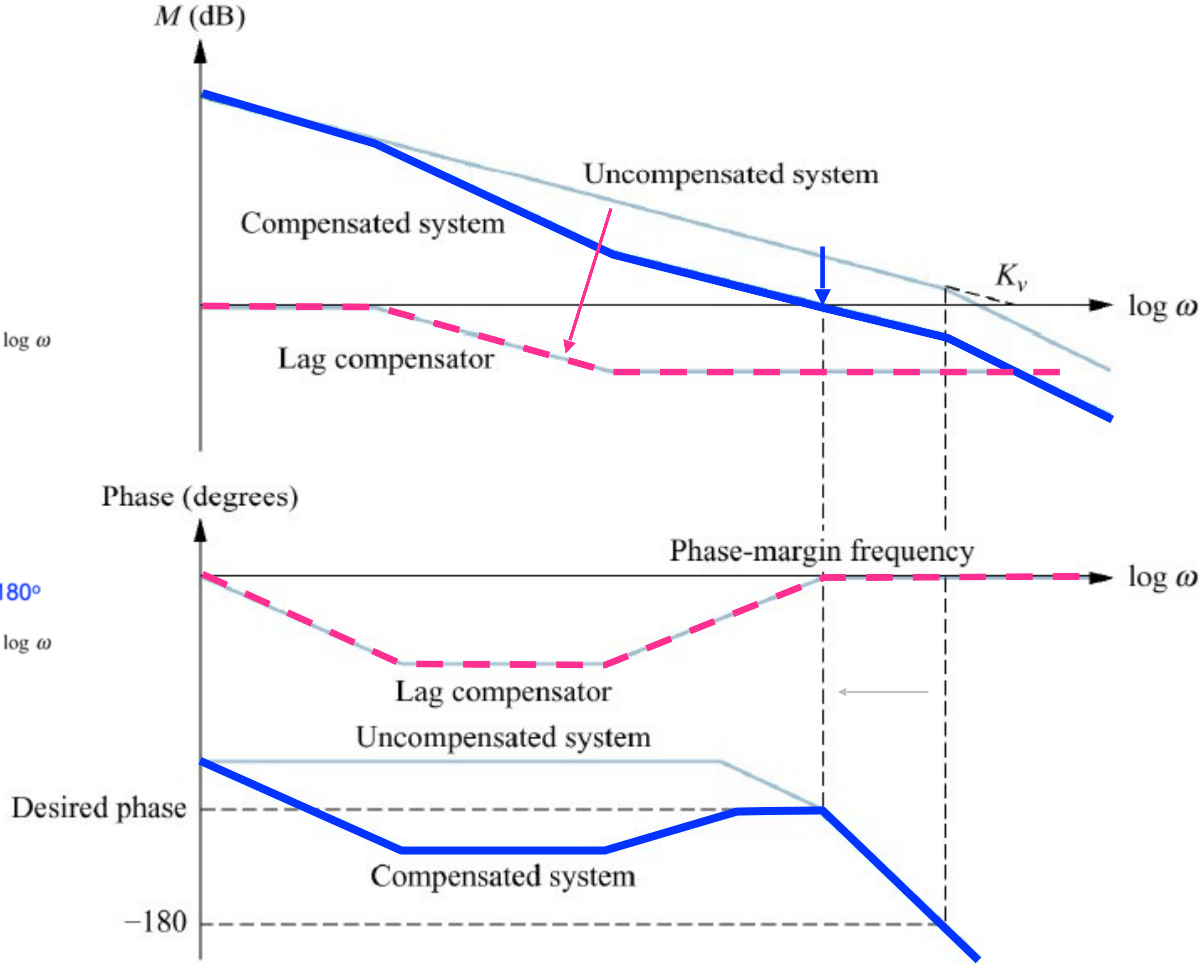
$$= \frac{3}{2}\cdot\frac{\left(\frac{s}{3}+1\right)}{s(s+1)\cdot\left(\frac{s}{2}+1\right)}$$



# Ex. Compensador de Atraso de Fase (*Lag*)

1. Melhorar constante de erro estático sem comprometer a estabilidade do sistema;
2. Aumentar a Margem de Fase do sistema de forma a satisfazer a resposta transitória desejada.



# Resumo:
# Caso de pólo simples real…

$$G(s) = 1 + \frac{s}{\omega_p} = \frac{1}{1 + j\frac{\omega}{\omega_p}}$$

Onde $\omega_p$ = freq. de corte ($-3$ dB neste ponto);

**Magnitude**:

$$|G(j\omega)| = \left| {}^1\!/\!\left(1 + j\frac{\omega}{\omega_p}\right) \right| = {}^1\!/\!\sqrt{1^2 + \left(\frac{\omega}{\omega_p}\right)^2}$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10}\left[ {}^1\!/\!\sqrt{1 + \left(\frac{\omega}{\omega_p}\right)^2} \right]$$

Quando $\omega << \omega_p$, teremos: $\omega/\omega_p \rightarrow 0$ e:

$$\sqrt{1 + ({}^{\omega}\!/\!_{\omega_p})^2} \approx 1 \;\therefore\; |G(j\omega)|_{dB} \approx -20\log_{10}(1/1) = 0$$

Quando $\omega >> \omega_p$, acontece: $\omega/\omega_p \rightarrow \infty$ e:

$$\sqrt{1 + ({}^{\omega}\!/\!_{\omega_p})^2} \approx \sqrt{({}^{\omega}\!/\!_{\omega_p})^2} \approx {}^{\omega}\!/\!_{\omega_p}$$

$$|G(j\omega)|_{dB} \approx -20\log_{10}(\omega/\omega_p)$$

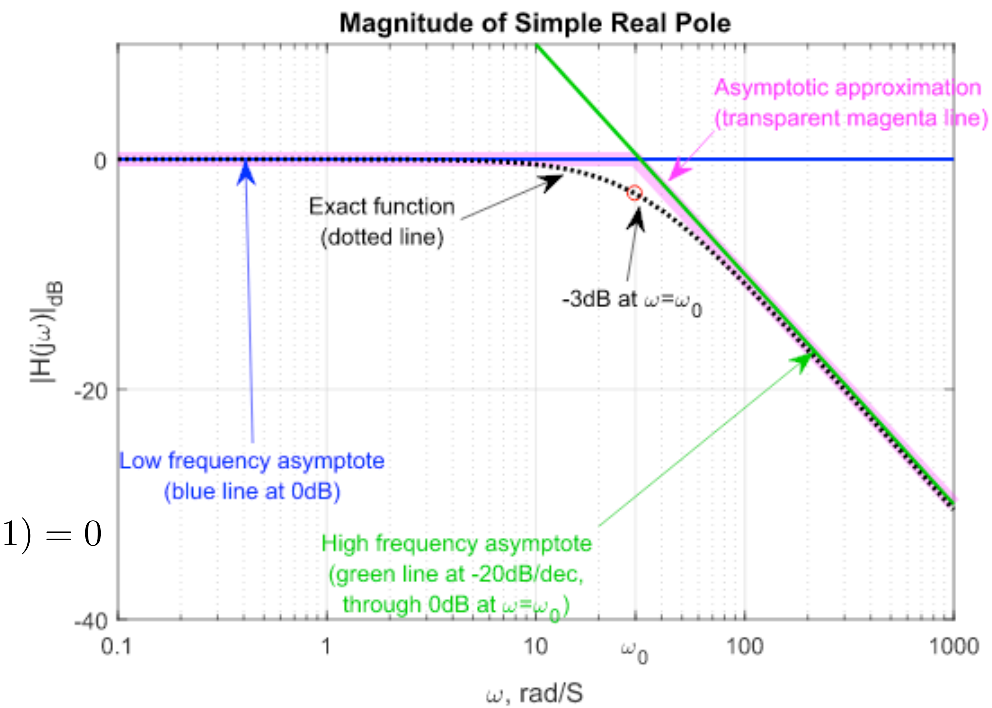
Se: $\omega = 10\,\omega_p \;\therefore\; |G(j\omega)|_{dB} = -20\log_{10}({}^{10\cancel{\omega_p}}\!/\!_{\cancel{\omega_p}})$

$$|G(j\omega)|_{dB} = -20\log_{10}(10) = -20 \text{ dB}$$

Quando $\omega = \omega_p$:

$$|G(j\omega_p)|_{dB} = 20\log_{10}\left( {}^1\!/\!\sqrt{1 + ({}^{\omega_p}\!/\!_{\omega_p})^2} \right)$$

$$|G(j\omega_p)|_{dB} = 20\log_{10}\left({}^1\!/\!\sqrt{2}\right) \approx -3 \text{ dB}$$



# Resumo:
# Caso de pólo simples real…

$$G(s) = 1 + \frac{s}{\omega_p} = \frac{1}{1 + j\frac{\omega}{\omega_p}}$$

Onde $\omega_p$ = freq. de corte ($-45^o$ neste ponto);

**Fase**:

$$\angle G(j\omega) = \angle \left[ {}^{1}\!/\!\left(1 + j\frac{\omega}{\omega_p}\right) \right]$$

$$\angle G(j\omega) = -\angle \left(1 + j\frac{\omega}{\omega_p}\right) = -\tan^{-1}\left(\frac{\omega}{\omega_p}\right)$$

Quando $\omega << \omega_p$, teremos: $\omega/\omega_p \to 0$ e:

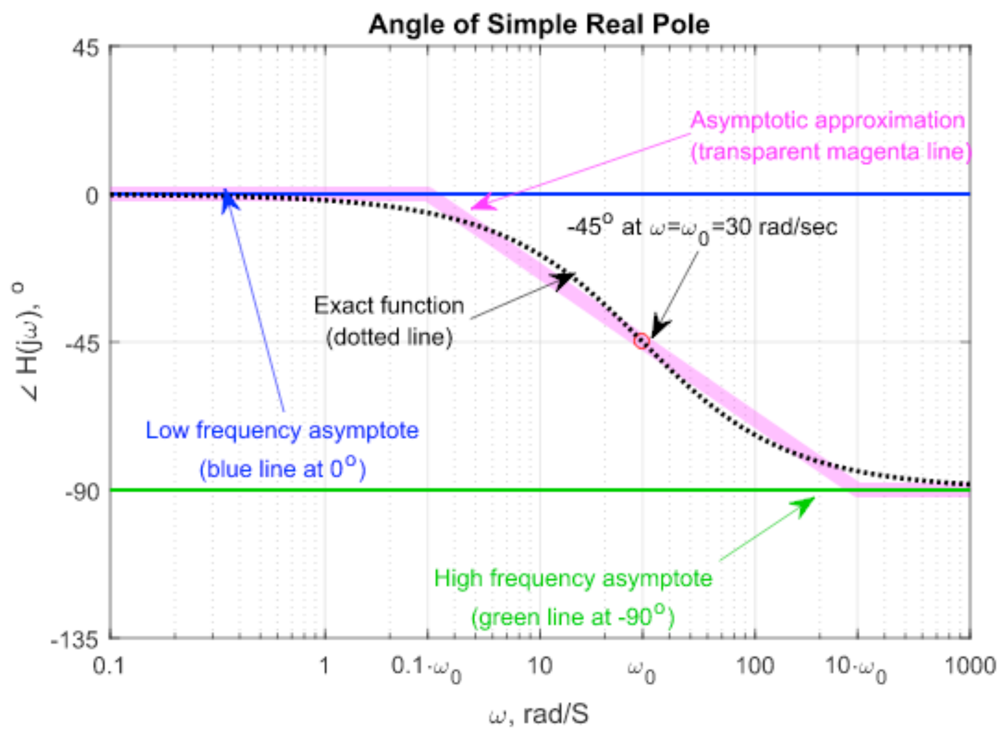
$$\angle G(j\omega) \approx -\tan^{-1}(0) = 0^o = 0 \text{ rad}$$

Quando $\omega >> \omega_p$, acontece: $\omega/\omega_p \to \infty$ e:

$$\angle G(j\omega) \approx -\tan^{-1}(\infty) = -90^o = -\frac{\pi}{2} \text{ rad}$$

Quando $\omega = \omega_p$:

$$\angle G(j\omega) \approx -\tan^{-1}(1) = -45^o = -\frac{\pi}{4} \text{ rad}$$




Ref.: https://lpsa.swarthmore.edu/Bode/BodeHow.html (acessado Jun/2022)

# Resumo:
# Caso de zero simples real…

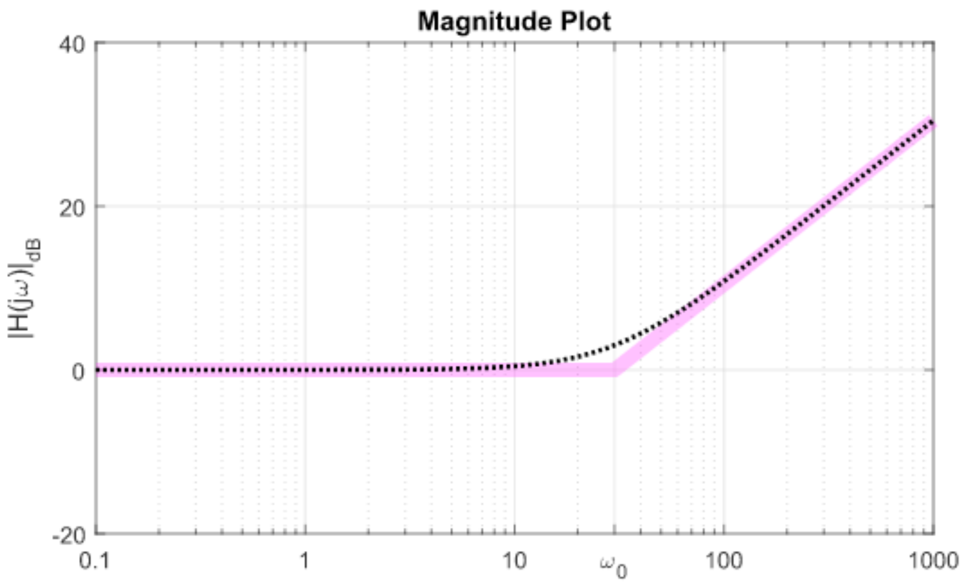
$$G(s) = 1 + \frac{s}{\omega_z} = 1 + j\frac{\omega}{\omega_z}$$

Onde $\omega_z$ = freq. de corte ($+3$ dB neste ponto);

**Magnitude**:

$$|G(j\omega)| = \left|1 + j\frac{\omega}{\omega_z}\right|$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10}\left[\sqrt{1 + \left(\omega/\omega_z\right)^2}\right]$$




Quando $\omega << \omega_z$, teremos: $\omega/\omega_z \to 0$ e:

$$\sqrt{1 + \left(\omega/\omega_p\right)^2} \approx 1 \;\therefore\; |G(j\omega)|_{dB} \approx 20\log_{10}(1) = 0$$

Quando $\omega >> \omega_z$, acontece: $\omega/\omega_z \to \infty$ e:

$$\sqrt{1 + \left(\omega/\omega_z\right)^2} \approx \sqrt{\left(\omega/\omega_z\right)^2} \approx \omega/\omega_z$$

$$|G(j\omega)|_{dB} \approx 20\log_{10}(\omega/\omega_z)$$

Se: $\omega = 10\,\omega_z \;\therefore\; |G(j\omega)|_{dB} = 20\log_{10}\left(10\cancel{\omega_z}/\cancel{\omega_z}\right)$

$$|G(j\omega)|_{dB} = 20\log_{10}(10) = 20 \text{ dB}$$

Quando $\omega = \omega_z$:

$$|G(j\omega_z)|_{dB} = 20\log_{10}\left(\sqrt{1 + \left(\omega_z/\omega_z\right)^2}\right)$$

$$|G(j\omega_p)|_{dB} = 20\log_{10}\left(\sqrt{2}\right) \approx 3 \text{ dB}$$

# Resumo:
# Caso de zero simples real...

$$G(s) = 1 + \frac{s}{\omega_z} = 1 + j\frac{\omega}{\omega_z}$$

Onde $\omega_z$ = freq. de corte ($+45^o$ neste ponto);

**Fase**:

$$\angle G(j\omega) = \angle\left(1 + j^{\omega}/_{\omega_p}\right)$$

$$\angle G(j\omega) = \angle\left(1 + j\frac{\omega}{\omega_p}\right) = \tan^{-1}\left(\frac{\omega}{\omega_p}\right)$$

Quando $\omega << \omega_z$, teremos: $\omega/\omega_z \rightarrow 0$ e:

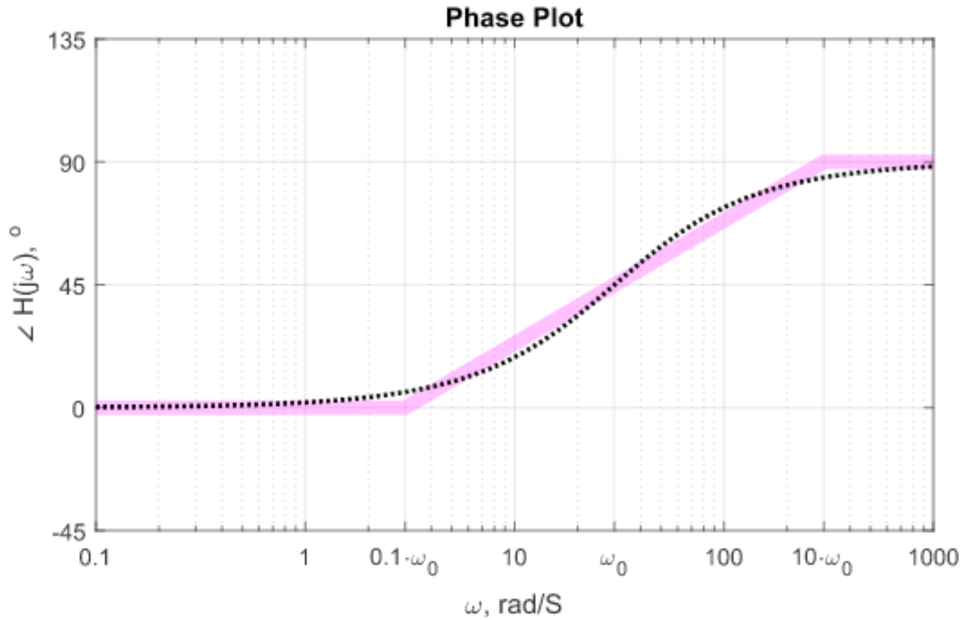
$$\angle G(j\omega) \approx \tan^{-1}(0) = 0^o = 0 \text{ rad}$$

Quando $\omega >> \omega_z$, acontece: $\omega/\omega_z \rightarrow \infty$ e:

$$\angle G(j\omega) \approx \tan^{-1}(\infty) = 90^o = \frac{\pi}{2} \text{ rad}$$

Quando $\omega = \omega_z$:

$$\angle G(j\omega) \approx \tan^{-1}(1) = 45^o = \frac{\pi}{4} \text{ rad}$$




Ref.: https://lpsa.swarthmore.edu/Bode/BodeHow.html (acessado Jun/2022)

# Resumo:
# Pólo na origem

$$G(s) = \frac{1}{s} \;\therefore\; G(j\omega) = \frac{1}{j\omega} = -\frac{j}{\omega}$$

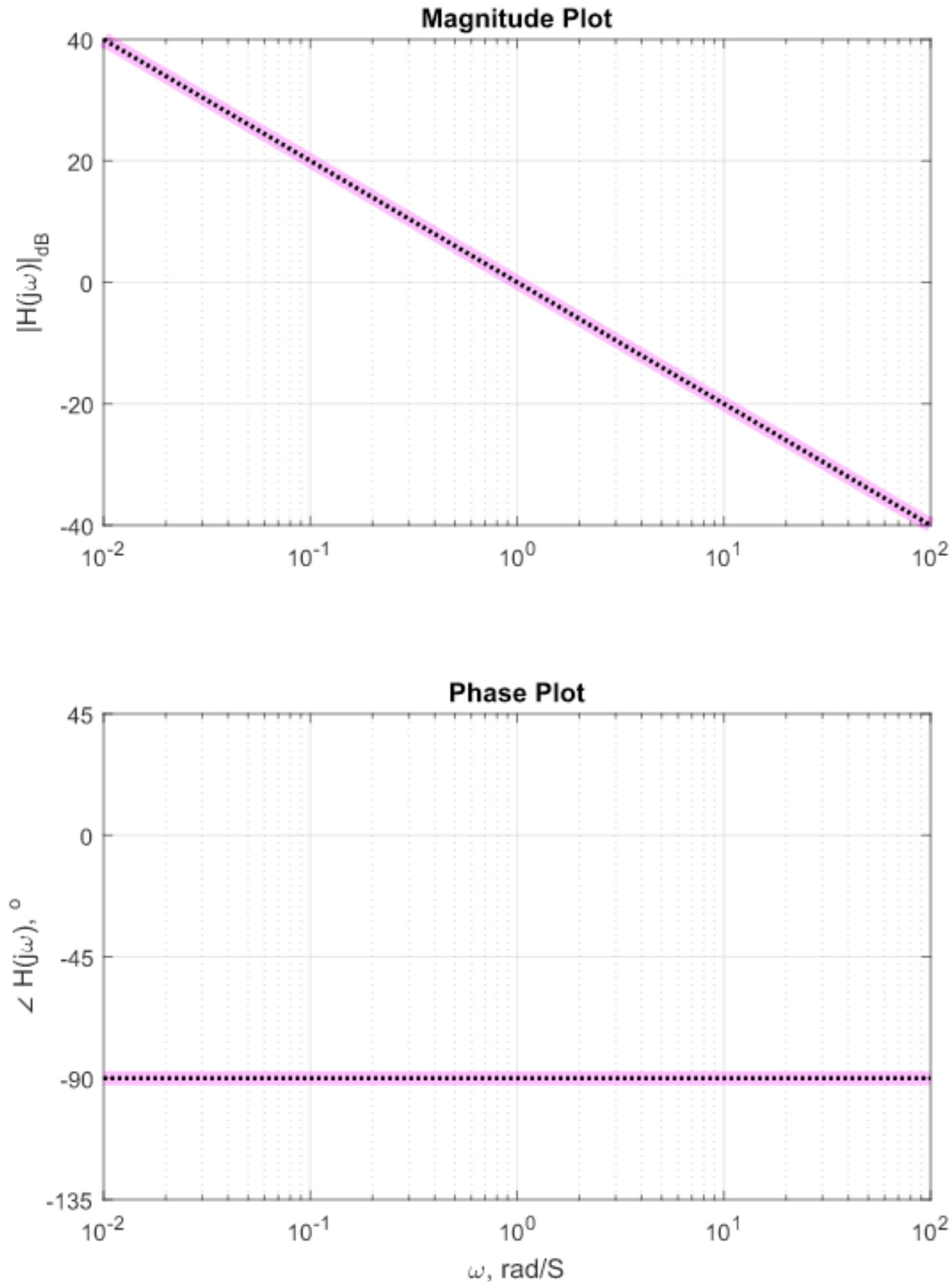
Magnitude:

$$|G(j\omega)| = \left|-\frac{j}{\omega}\right| = \frac{1}{\omega}$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10}\left(\frac{1}{\omega}\right) = -20\log_{10}(\omega)$$

Fase:

$$\angle G(j\omega) = \angle\left(-\frac{j}{\omega}\right) = -90^o$$



# Resumo:
# Zero na origem

$$G(s) = s \;\; \therefore \;\; G(j\omega) = j\omega$$

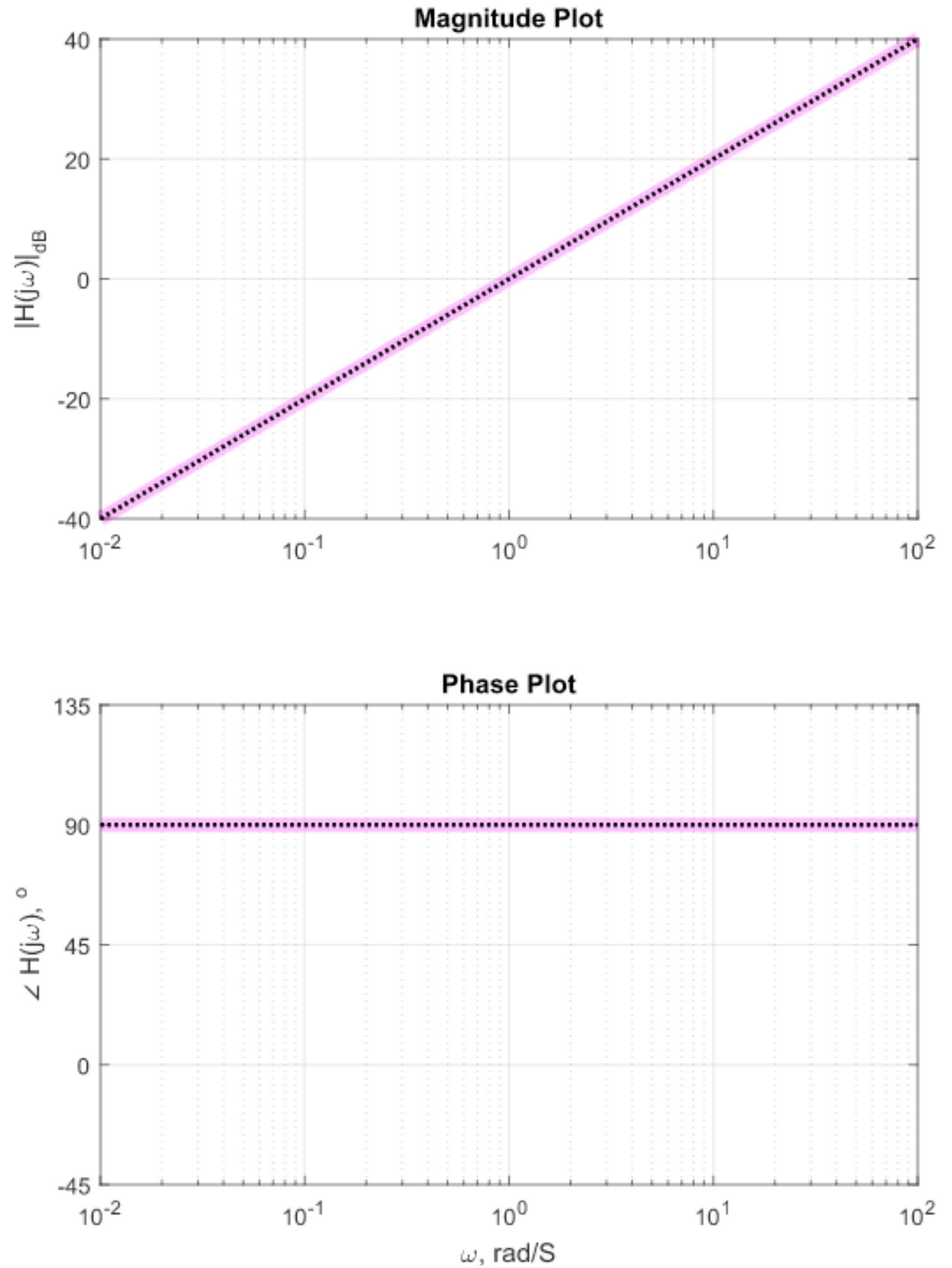
Magnitude:

$$|G(j\omega)| = \left| j\omega \right| = \omega$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10}(\omega) = 20\log_{10}(\omega)$$

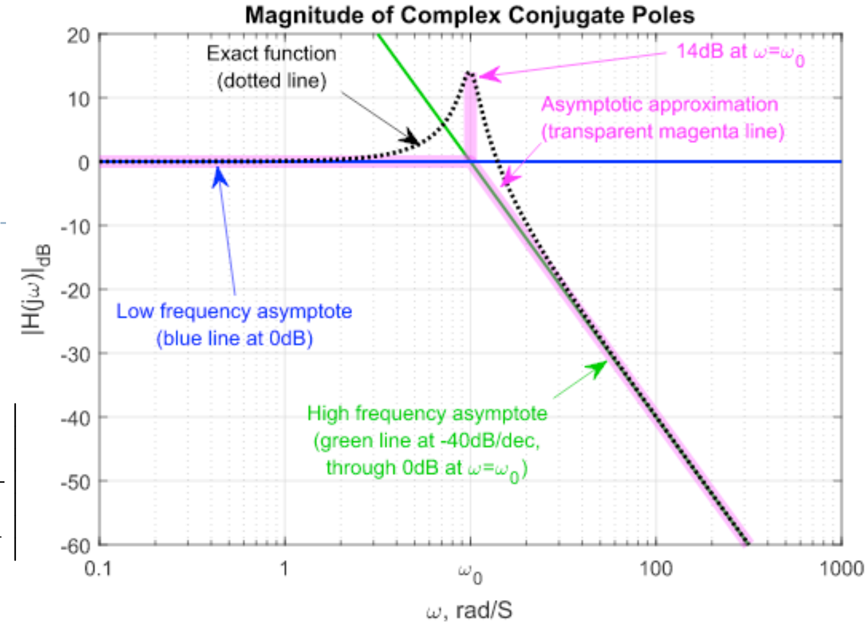
Fase:

$$\angle G(j\omega) = \angle\left(j\omega\right) = 90^o$$



# Resumo:
# Pólos complexos

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + s2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$




Magnitude:

$$|G(j\omega)| = \left|\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1}\right| = \left|\frac{1}{-\left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right) + 1}\right|$$

$$|G(j\omega)| = \left|\frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + j\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]}\right| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\,\frac{\omega}{\omega_n}\right)^2}}$$

$$|G(j\omega)|_{dB} = -20 \cdot \log_{10}\left\{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\,\frac{\omega}{\omega_n}\right)^2}\right\} \quad (1)$$

Caso de $\omega << \omega_n$: pode-se aproximar (1) para:

$$|G(j\omega)|_{dB} = -20 \cdot \log_{10}(1) = 0$$

Caso de $\omega >> \omega_n$: pode-se aproximar (1) para:

$$|G(j\omega)|_{dB} = -20 \cdot \log_{10}\left[\left(\frac{\omega}{\omega_n}\right)^2\right] = -40\log_{10}\left(\frac{\omega}{\omega_n}\right)$$

Isto implica que para década acima de $\omega_n$, magnitude $= -40$ (dB/déc).

Caso de $\omega \approx \omega_n$: ocorre um pico próximo de $\omega_n$:

Ocorre em $\omega_\tau$: $\omega_\tau = \omega_n\sqrt{1 - 2\zeta^2}$

Pico ocorre para: $0 < \zeta < 1/2$, com valor de:

$$|G(j\omega_\tau)| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

Ver próximo slide para maiores detalhes.

# Resumo
# Pólos complexos $(0 < \zeta < 1)$

Seja:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + s2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\,\frac{\omega}{\omega_n}\right)^2}}$$

Um pico de ganho pode ser identificado em $|G(j\omega)|$, fazendo:

$$\frac{d}{d\omega}\left(|G(j\omega)|\right)\Big|_{\omega\to 0}$$

Este pico ocorre em: $\omega_\tau = \omega_n\sqrt{1 - 2\zeta^2}$,

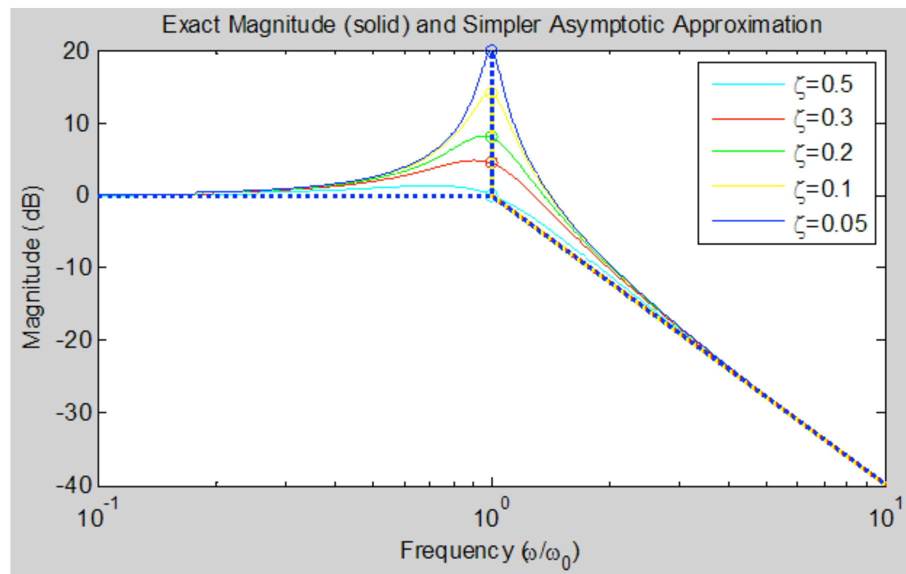
Com magnitude de:

$$|G(j\omega)| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \approx \frac{1}{2\zeta}$$

$$|G(j\omega)| = -20\log_{10}\left(2\zeta\sqrt{1 - \zeta^2}\right)$$

O pico apenas existe para: $0 < \zeta < 0{,}707\ (= 1/\sqrt{2})$

| ζ | **Peak frequency** | | | **Peak Height** | | |
|---|---|---|---|---|---|---|
| | Exact | Approx | % diff | Exact | Approx | % diff |
| | $\omega_0\sqrt{1-2\zeta^2}$ | $\omega_0$ | | $\frac{1}{2\zeta\sqrt{1-\zeta^2}}$ | $\frac{1}{2\zeta}$ | |
| 0.5 | $0.71\omega_0$ | $\omega_0$ | 29% | 1.15 | 1.00 | 15% |
| 0.4 | $0.83\omega_0$ | $\omega_0$ | 17% | 1.37 | 1.25 | 9.1% |
| 0.3 | $0.91\omega_0$ | $\omega_0$ | 9.5% | 1.75 | 1.67 | 4.8% |
| 0.2 | $0.96\omega_0$ | $\omega_0$ | 4.1% | 2.55 | 2.50 | 2.1% |
| 0.1 | $0.99\omega_0$ | $\omega_0$ | 1.0% | 5.02 | 5.00 | 0.5% |
| 0.05 | $1.00\omega_0$ | $\omega_0$ | 0.3% | 10.0 | 10.0 | 0.1% |




Ref.: https://lpsa.swarthmore.edu/Bode/underdamped/underdampedApprox.html#Simpler (acessado em Jun/2022)

# Resumo:
# Pólos complexos

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + s2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

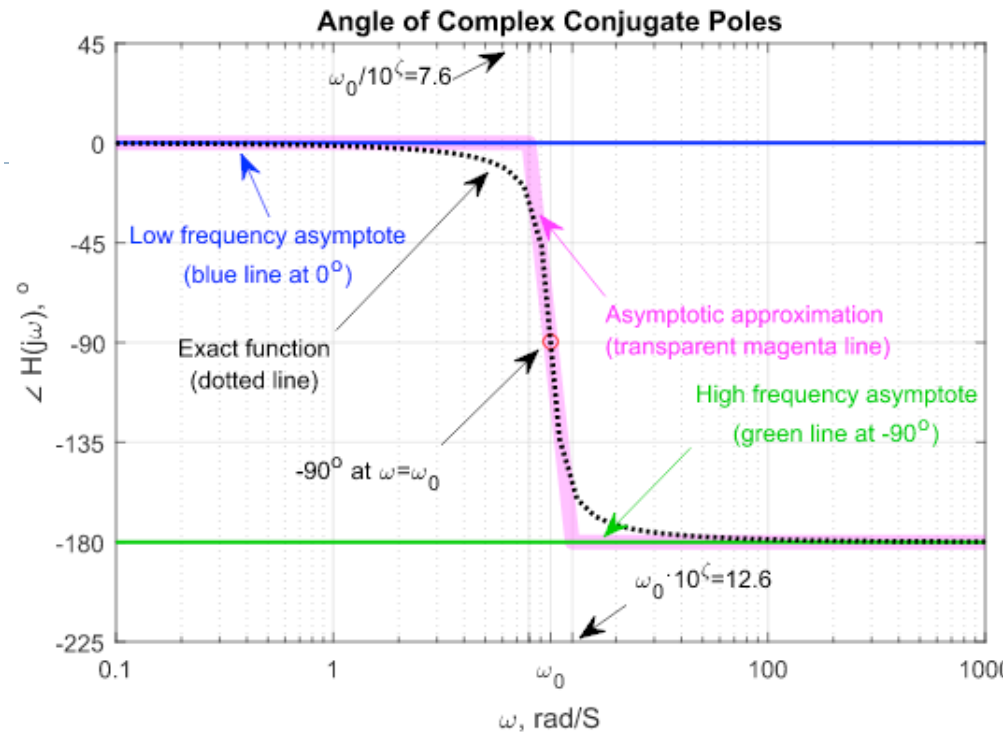
Fase:

$$\angle G(j\omega) = \angle\left[\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1}\right]$$

$$\angle G(j\omega) = -\angle\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1\right]$$

$$\angle G(j\omega) = -\angle\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right)\right]$$

$$\angle G(j\omega) = -\tan^{-1}\left[\left(2\zeta\frac{\omega}{\omega_n}\right) \Big/ \left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)\right] \quad (2)$$




Caso de $\omega << \omega_n$: pode-se aproximar (2) para:

$$\angle G(j\omega) \approx -\tan^{-1}\left(\frac{2\zeta\omega}{\omega_n}\right) \approx -\tan^{-1}(0) = 0^o = 0 \text{ rad}$$

Caso de $\omega >> \omega_n$: pode-se aproximar (2) para:

$$\angle G(j\omega) \approx -180^o = -\pi \text{ rad}$$

Caso de $\omega = \omega_n$:

$$\angle G(j\omega) \approx -90^o = -\pi/2 \text{ rad}$$

# Resumo
# Pólos complexos $(0 < \zeta < 1)$

Seja:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + s2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

$$\angle G(j\omega) = -\tan^{-1}\left[\left(2\zeta\frac{\omega}{\omega_n}\right) \Big/ \left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)\right] \quad (2)$$

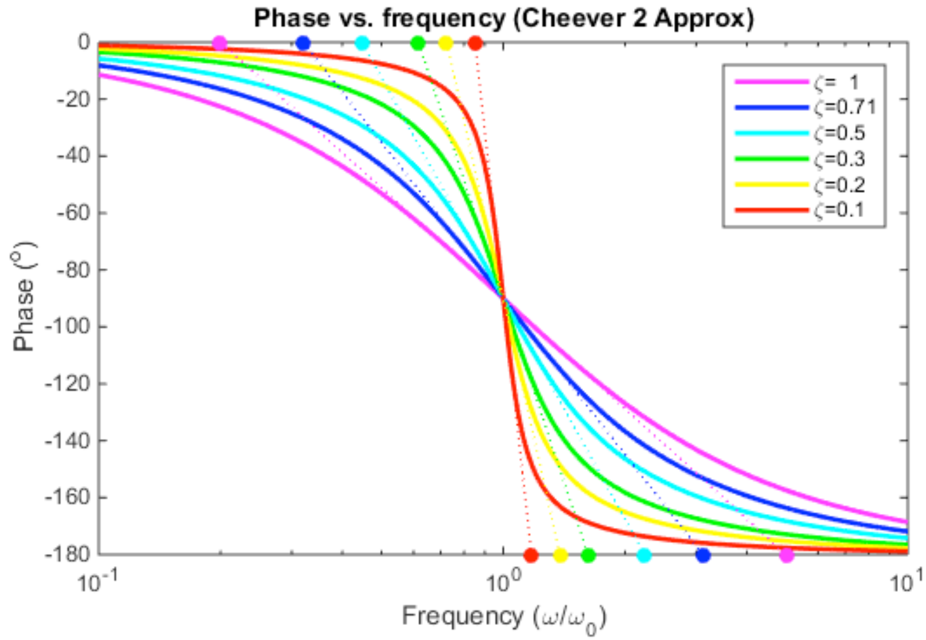
A declividade de $\angle G(j\omega)$, pode ser obtida, fazendo:

$$\frac{d}{d\omega}\left(\angle G(j\omega)\right)\Big|_{\omega \to 0}$$

O que permite identificar os pontos:

$$\omega_{low} = \omega_n e^{-\frac{\pi}{2}\zeta} \approx \omega_n \cdot 0{,}2^{\zeta}$$

$$\omega_{high} = \frac{\omega_n}{e^{-\frac{\pi}{2}\zeta}} \approx \frac{\omega_n}{0{,}2^{\zeta}} = \omega \cdot 5^{\zeta}$$




Ref.: https://lpsa.swarthmore.edu/Bode/underdamped/underdampedApprox.html#Simpler (acessado em Jun/2022)

# Resumo: Esboços Diagramas de Bode

- https://lpsa.swarthmore.edu/Bode/BodeHow.html

The Asymptotic Bode Diagram: Deri… ×

https://lpsa.swarthmore.edu/Bode/BodeHow.html

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## Interactive Demo: Bode Plot of a Pair of Complex Conjugate Poles

This demonstration shows how a second order pole (complex conjugate roots) expressed as:

$$H(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1} = \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j2\zeta\frac{\omega}{\omega_0}},$$

is displayed on a Bode plot. You can change $\omega_0$, and $\zeta$. The value of $\omega_0$ is constrained such that $0.1 \le \omega_0 \le 10$ rad/second, and $0.05 \le \zeta \le 0.99$.

Enter value for $\omega_0$: 1.000 or click and drag on graph to set $\omega_0$.,

and use text-box or slider, below, for $\zeta$.

0.200 $\zeta$ 0.05 — 0.99

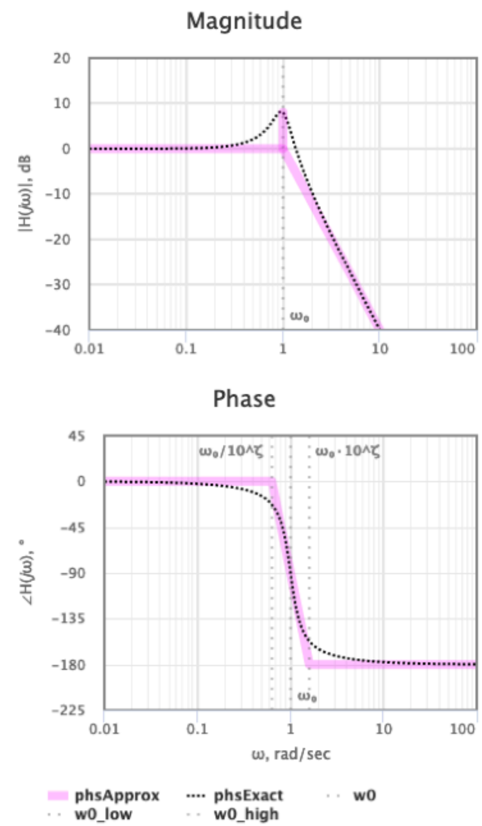
**Asymptotic Magnitude:** The asymptotic magnitude plot starts (at low frequencies) at 0 dB and stays at that level until it gets to $\omega_0$. At that point the gain starts dropping with a slope of -40 dB/decade. *Note: it is -40 dB per decade because there are two poles in the denominator.*

If $\zeta<0.5$ we estimate the peak height as $|H(j\omega_{peak})| \approx \frac{1}{2\zeta}$ (exact height is $|H(j\omega_{peak})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$). We approximate the peak location at $\omega_{peak} \approx \omega_0$ (exact peak location is at $\omega_{peak} = \omega_0\sqrt{1-2\zeta^2}$ ).

However, if $\zeta \ge 0.5$, the peak is sufficiently small that we don't include it in our plot.

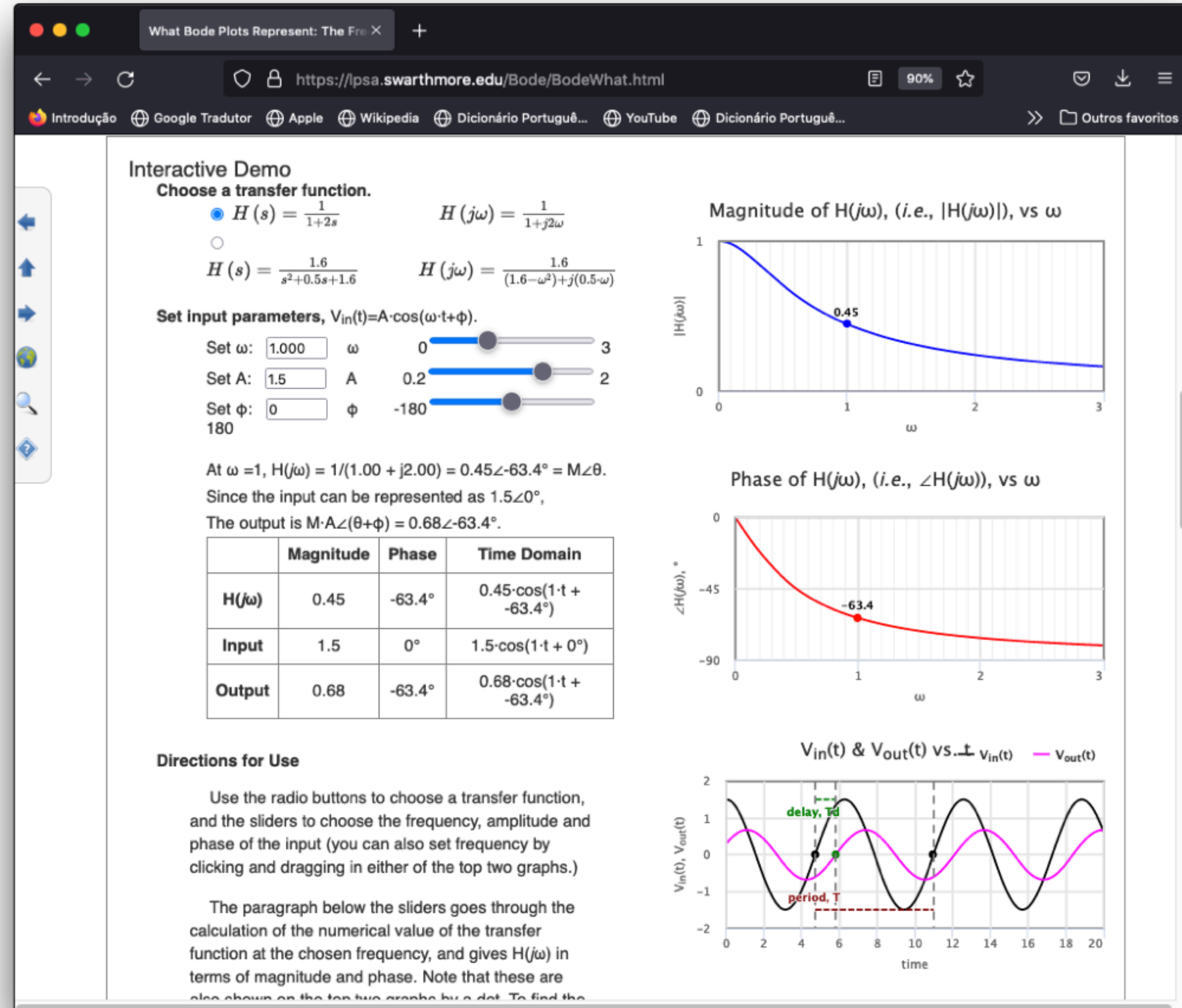
Since $\zeta<0.5$, we draw a peak. Note how close together the approximate and exact values are for $\omega_{peak}$ and $|H(j\omega_{peak})|$.

| | $\omega_{peak}$ | $\lvert H(j\omega_{peak})\rvert$ | $\lvert H(j\omega_{peak})\rvert_{dB}$ |
|---|---|---|---|
| **Approximate** | 1.00 | 2.50 | 7.96 |
| **Exact** | 0.96 | 2.55 | 8.14 |

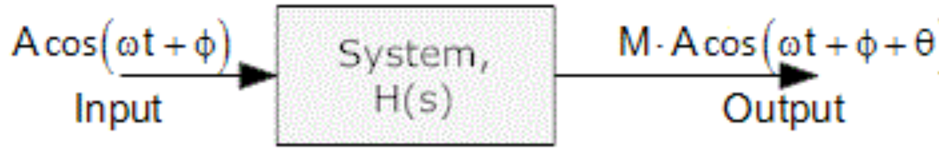


# Resposta frequencial:: Aplicação

- https://lpsa.swarthmore.edu/Bode/BodeWhat.html



# Resposta frequencial
# Aplicação




- Problema: mostre o sinal de saída para o sistema ao lado quando se aplica ao mesmo um cosseno de 2 Vpp à frequência de 1 Hz.
- Considerar: R = 2 MΩ e C = 1 μF
- Solução:

$$H(s) = \frac{1}{1+sRC} = \frac{1}{1+2\times 10^{6}\cdot 1\times 10^{-6}\cdot s} = \frac{1}{1+2s}$$

$$H(j\omega) = \frac{1}{1+j2\omega} = \frac{1}{1+j\dfrac{\omega}{\omega_p}}$$

$$|H(j\omega)| = \left| 1/\left(1+j\frac{\omega}{\omega_p}\right)\right| = 1/\sqrt{1^2+\left(\frac{\omega}{\omega_p}\right)^2}$$

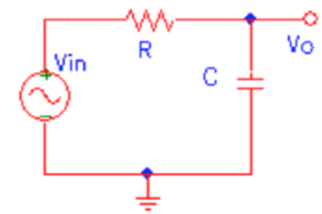
$$\angle H(j\omega) = -\angle\left(1+j\frac{\omega}{\omega_p}\right) = -\tan^{-1}\left(\frac{\omega}{\omega_p}\right)$$

$\omega_p = 1/2$ (rad/s)

$$\therefore f_p = \frac{\omega}{2\pi} = \frac{1}{4\pi} = 0{,}079577 \text{ Hz}$$

$$V_{out}(j\omega) = V_{in}(j\omega)\cdot H(j\omega)$$

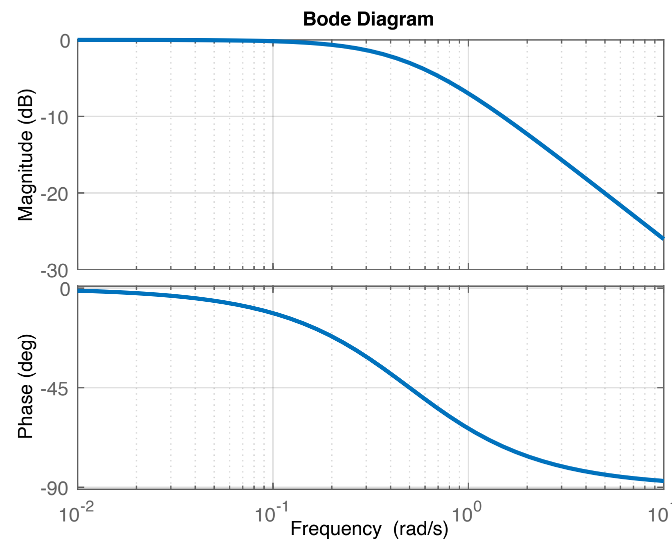
$$|V_{out}(j\omega)| =$$

| Circuit | Transfer Function |
|---|---|
|  | $H(s) = \dfrac{V_{out}(s)}{V_{in}(s)} = \dfrac{1}{1+sRC}$ |

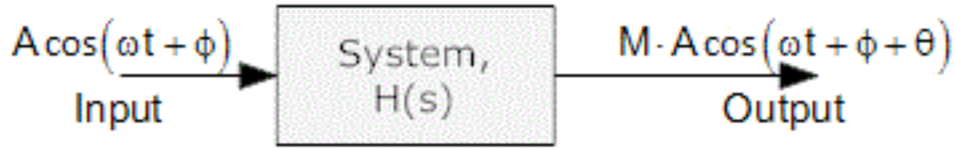


Usando Matlab:

```
>> H=tf(1,[2 1])
H =
      1
   -------
   2 s + 1
>> pole(H)
          -0.5
>> f_p=1/(4*pi)
f_p =
      0.079577
>> zpk(H)
    0.5
   -------
   (s+0.5)
>> bode(G)
```



# Resposta frequencial
# Aplicação




- Problema: mostre o sinal de saída para o sistema ao lado quando se aplica ao mesmo um cosseno de 2 Vpp à frequência de 1 Hz.
- Considerar: R = 2 MΩ e C = 1 μF
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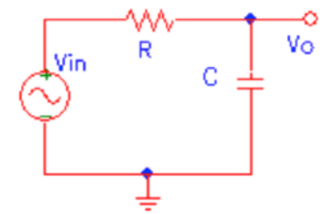
| Circuit | Transfer Function |
|---|---|
|  | $H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1+sRC}$ |

$$H(s) = \frac{1}{1+sRC} = \frac{1}{1+2\times10^6\cdot1\times10^{-6}\cdot s} = \frac{1}{1+2s}$$

$$H(j\omega) = \frac{1}{1+j2\omega} = \frac{1}{1+j\frac{\omega}{\omega_p}}$$

$$|H(j\omega)| = \left|{}^{1}\!/\left(1+j\frac{\omega}{\omega_p}\right)\right| = {}^{1}\!/\sqrt{1^2+\left(\frac{\omega}{\omega_p}\right)^2}$$

$$\angle H(j\omega) = -\angle\left(1+j\frac{\omega}{\omega_p}\right) = -\tan^{-1}\left(\frac{\omega}{\omega_p}\right)$$

$\omega_p = 1/2$ (rad/s)

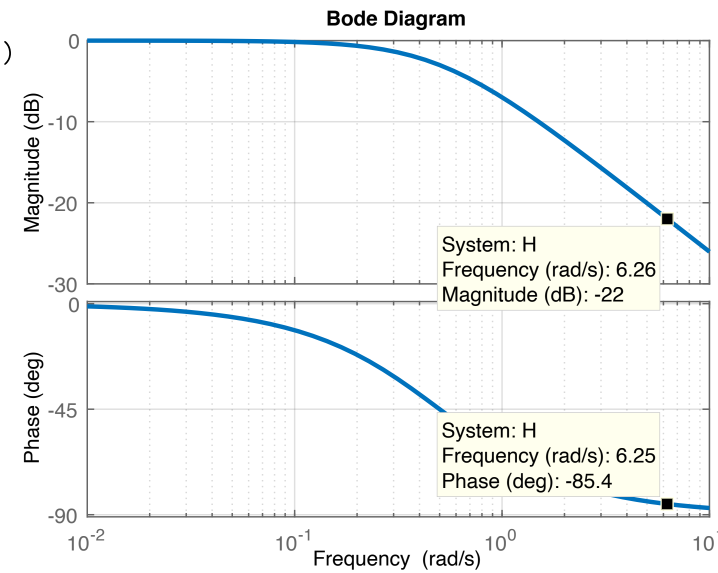
$$\therefore f_p = \frac{\omega}{2\pi} = \frac{1}{4\pi} = 0{,}079577 \text{ Hz}$$

$$V_{out}(j\omega) = V_{in}(j\omega)\cdot H(j\omega)$$

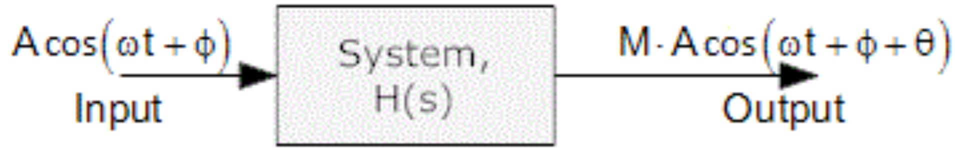
$$|V_{out}(j\omega)| =$$

Usando Matlab:

```
>> H=tf(1,[2 1])
H =
      1
   -------
   2 s + 1
>> pole(H)
           -0.5
>> f_p=1/(4*pi)
f_p =
        0.079577
>> zpk(H)
    0.5
   -------
   (s+0.5)
>> bode(G)
>> f=2*pi
f =
          6.2832
```



# Resposta frequencial
# Aplicação




- Problema: mostre o sinal de saída para o sistema ao lado quando se aplica ao mesmo um cosseno de 2 Vpp à frequência de 1 Hz.
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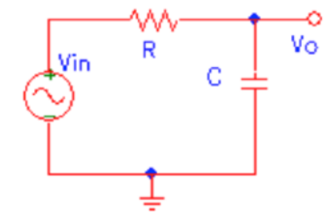
$$H(s) = \frac{1}{1+sRC} = \frac{1}{1+2\times 10^{6}\cdot 1\times 10^{-6}\cdot s} = \frac{1}{1+2s}$$

$$H(j\omega) = \frac{1}{1+j2\omega} = \frac{1}{1+j\frac{\omega}{\omega_p}}$$

| Circuit | Transfer Function |
| --- | --- |
|  | $H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1+sRC}$ |

Ganho em $\omega = 1$ Hz $= 2\pi$ rad/s:

$|G(j\omega)|_{dB} = -22$ dB

$-22 = 20\log_{10}(|G(j\omega)|)$

$|G(j\omega)| = 10^{-\frac{22}{20}} = 0{,}079433$

$\angle G(j\omega) = -85{,}4^o = -85{,}4^o \cdot \frac{\pi}{180^o} = -1.4905$ rad

$T \rightarrow 360^o\,(2\pi)$

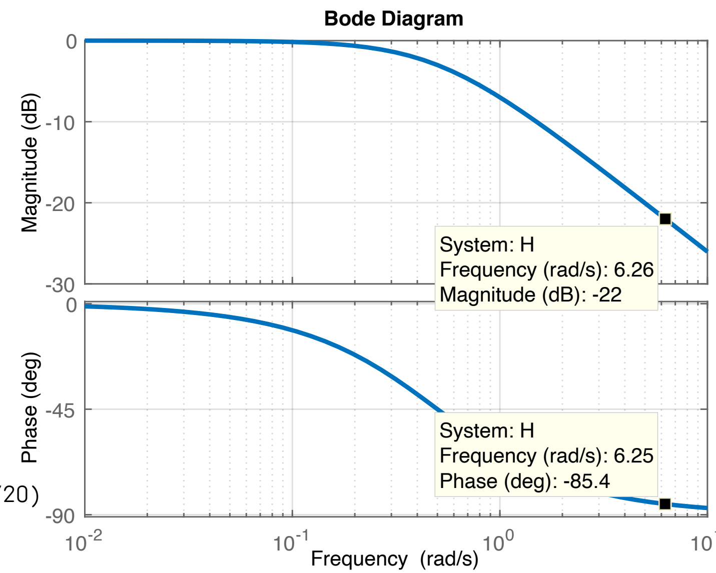
$\Delta t \leftarrow 85{,}4^o$

$\Delta t = \frac{85{,}4^o \cdot T}{360^o} = 0{,}037755$ segundos

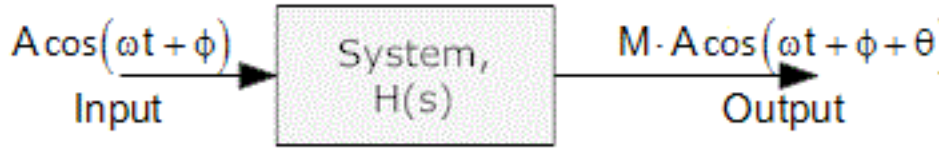
Considerando $T = 0{,}15915$ segundos.

Usando Matlab:

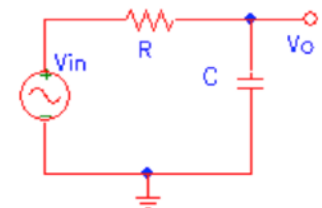
```
>> H=tf(1,[2 1])
H =
      1
   -------
   2 s + 1
>> pole(H)
          -0.5
>> f_p=1/(4*pi)
f_p =
        0.079577
>> zpk(H)
    0.5
   -------
   (s+0.5)
>> bode(G)
>> f=2*pi
f =
          6.2832
>> ganho=10^(-22/20)
ganho =
        0.079433
>> T=1/f
T =
         0.15915
>> Delta_t=(85.4*T)/360
Delta_t =
        0.037755
```



# Resposta frequencial Aplicação



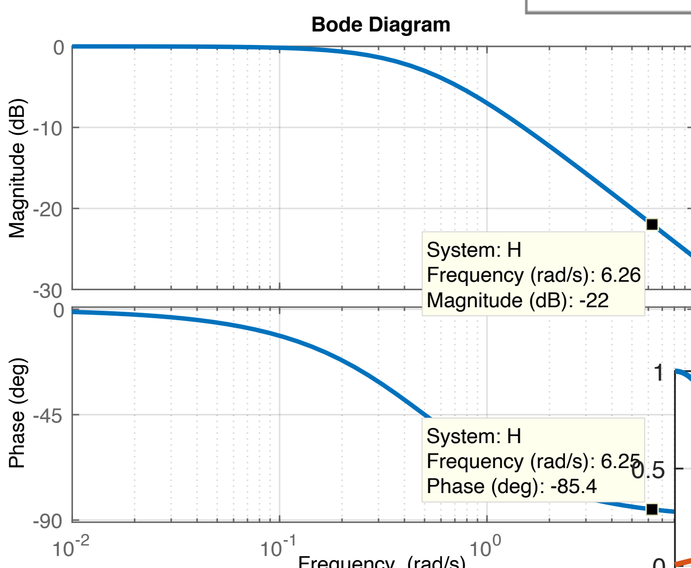

- Problema: mostre o sinal de saída para o sistema ao lado quando se aplica ao mesmo um cosseno de 2 Vpp à frequência de 1 Hz.
- Considerar: R = 2 MΩ e C = 1 µF
- Solução:

| Circuit | Transfer Function |
| --- | --- |
|  | $H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + sRC}$ |



Usando Matlab:

```
>> H=tf(1,[2 1])
H =
      1
   -------
   2 s + 1
>> pole(H)
            -0.5
>> f_p=1/(4*pi)
f_p =
     0.079577
>> zpk(H)
    0.5
   -------
   (s+0.5)
>> bode(G)
>> f=2*pi
f =
       6.2832
>> ganho=10^(-22/20)
ganho =
     0.079433
>> T=1/f
T =
      0.15915
>> Delta_t=(85.4*T)/360
Delta_t =
     0.037755
```




```
>> rad=(85.4*pi)/180
rad =
       1.4905
>> ezplot(@(t)1*cos(2*pi*1*t),[0 2]) %
onda entrada
>> hold on
>> ezplot(@(t)ganho*cos(2*pi*1*t-rad),
[0 2]) // onda saída
>> axis([0 2 -1 1])
>> grid
>> legend('V_{in}','V_{out}')
```