

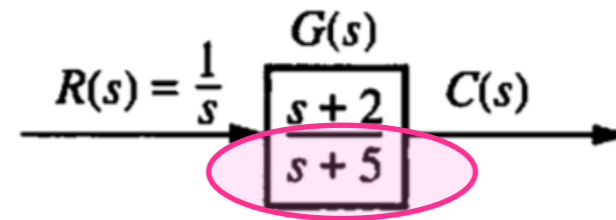
# ROOT LOCUS

INTRO...

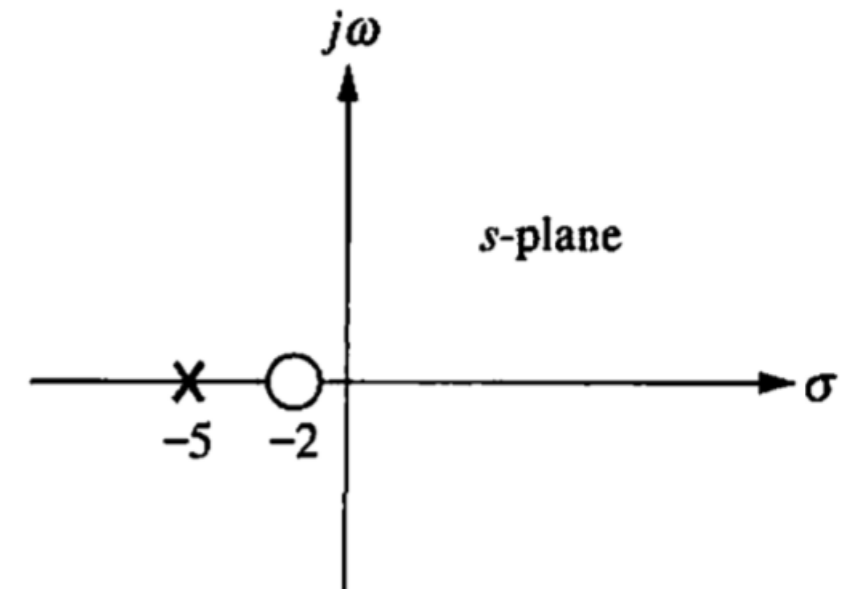
*Controle Automático II*  
*Prof. Fernando Passold*  
*Engenharia Elétrica — UPF*

# POLOS E ZEROS DE UM SISTEMA

- ▶ Seja o seguinte sistema:
- ▶ A resposta a uma entrada degrau unitário leva à:



(a) Sistema em MA.



(b) Pólos e zeros de MA no plano-s

Degrau:  $\mathcal{L}\{u(t)\} = 1/s$

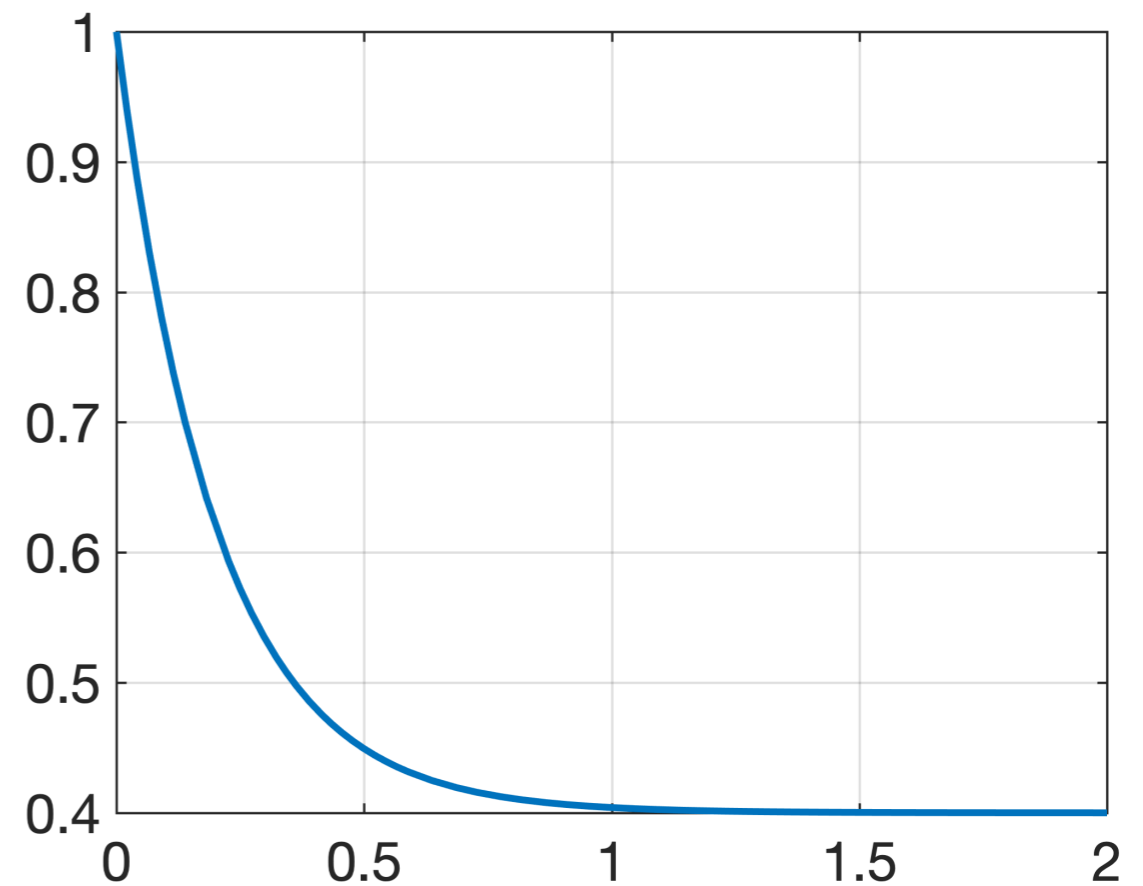
$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

$$A = \left. \frac{(s+2)}{(s+5)} \right|_{s=0} = \frac{2}{5}$$

$$B = \left. \frac{(s+2)}{s} \right|_{s=-5} = \frac{3}{5}$$

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

```
>> fplot(@(t) (2/5)+(3/5)*exp(-5*t), [0 2])
>> grid
```



ps.: Note que o pólo em  $s = -5$  é estável (resposta converge no tempo)

# POLOS E ZEROS DE UM SISTEMA

- ▶ Seja o seguinte sistema:
- ▶ A resposta a uma entrada degrau unitário leva à:

Degrau:  $\mathcal{L}\{u(t)\} = 1/s$

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} +$$

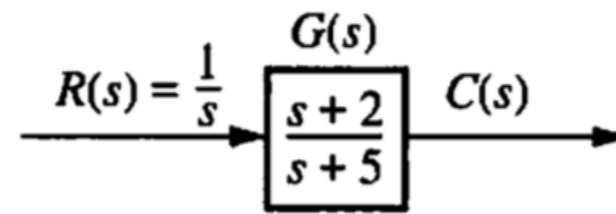
$$A = \left. \frac{(s+2)}{(s+5)} \right|_{s=0} = \frac{2}{5}$$

$$B = \left. \frac{(s+2)}{s} \right|_{s=-5} = \frac{3}{5}$$

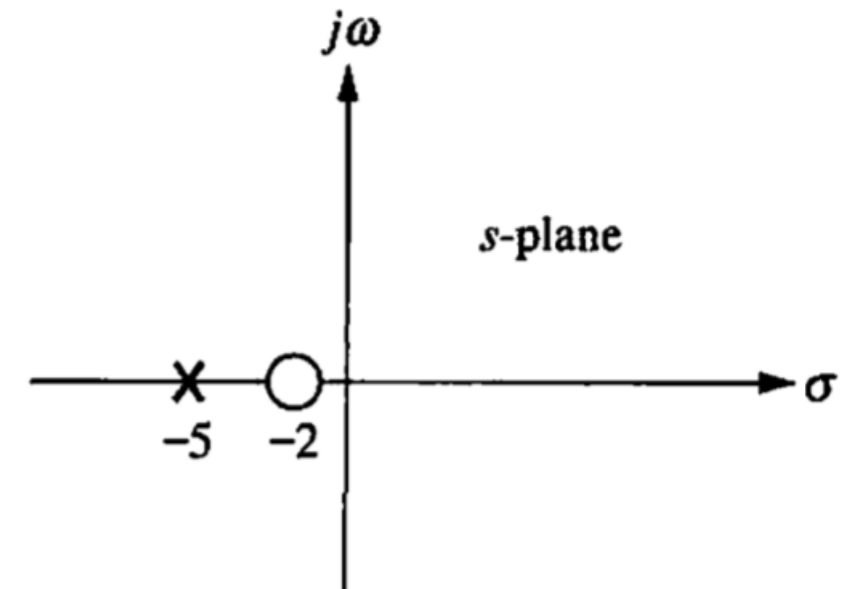
$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

zeros  $\rightarrow \times$

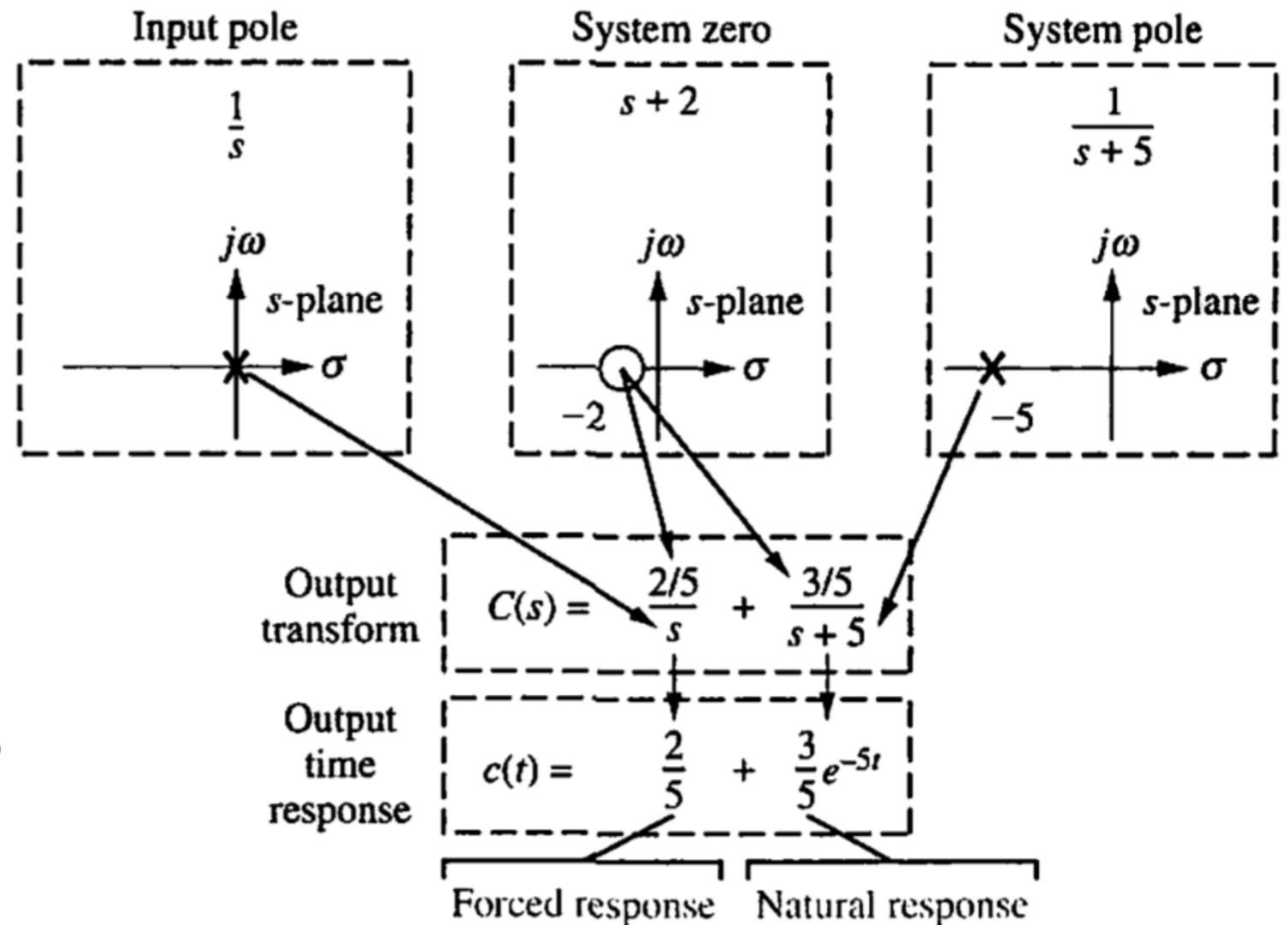
polos  $\rightarrow \circ$



(a) Sistema em MA.



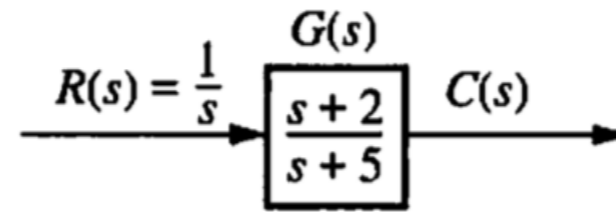
(b) Pólos e zeros de MA no plano-s



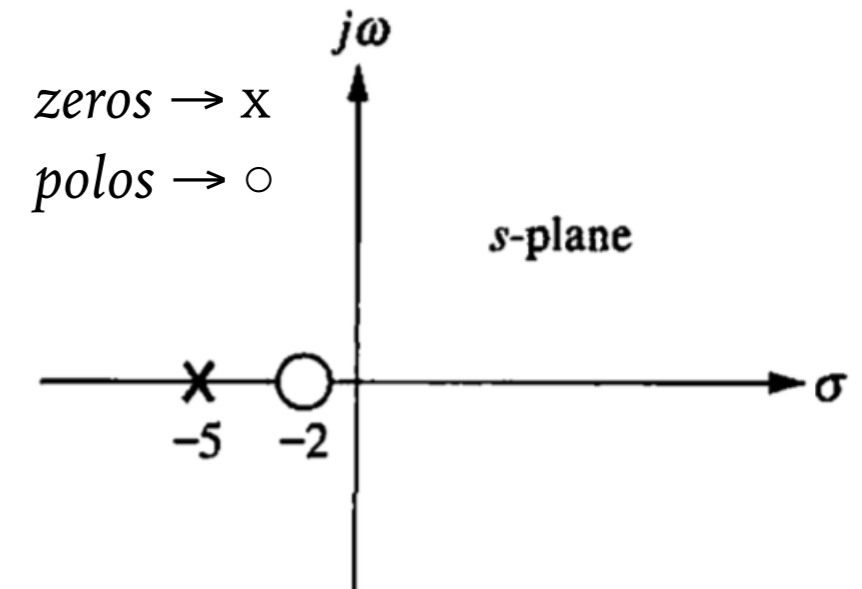
# POLOS E ZEROS DE UM SISTEMA

- Seja o seguinte sistema:
- A resposta a uma entrada degrau unitário leva à:
- O que acontece quando fechamos a malha com controlador proporcional?

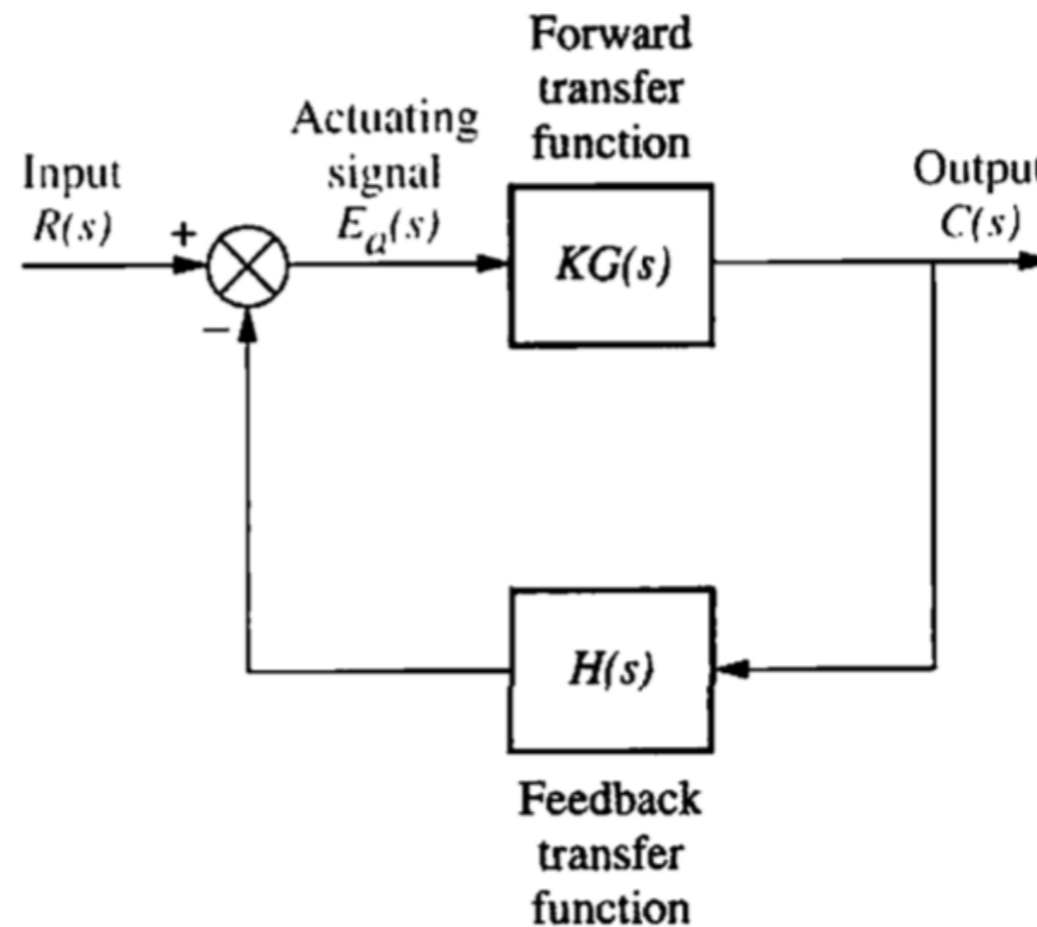
Vamos perceber que a medida que aumentamos  $K$ , os pólos em MF mudam de local.



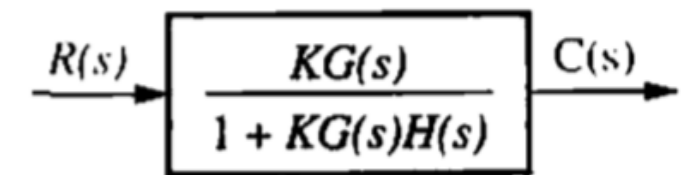
(a) Sistema em MA.



(b) Pólos e zeros de MA no plano-s



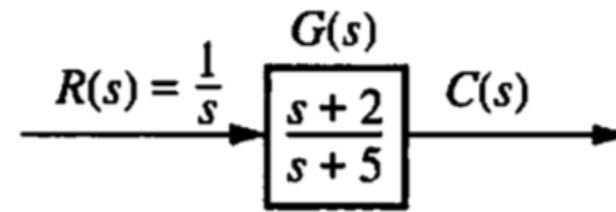
(c) Fechando a malha.



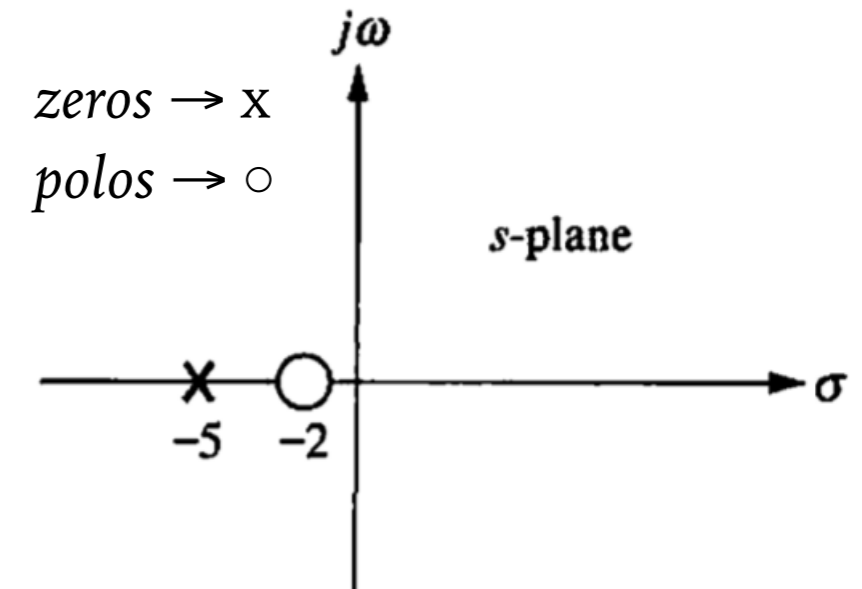
(d) Sistema equivalente (FTMF).

# POLOS E ZEROS DE UM SISTEMA

- Seja o seguinte sistema:
- A resposta a uma entrada degrau unitário leva à:
- O que acontece quando fechamos a malha com controlador proporcional



(a) Sistema em MA.



(b) Pólos e zeros de MA no plano-s

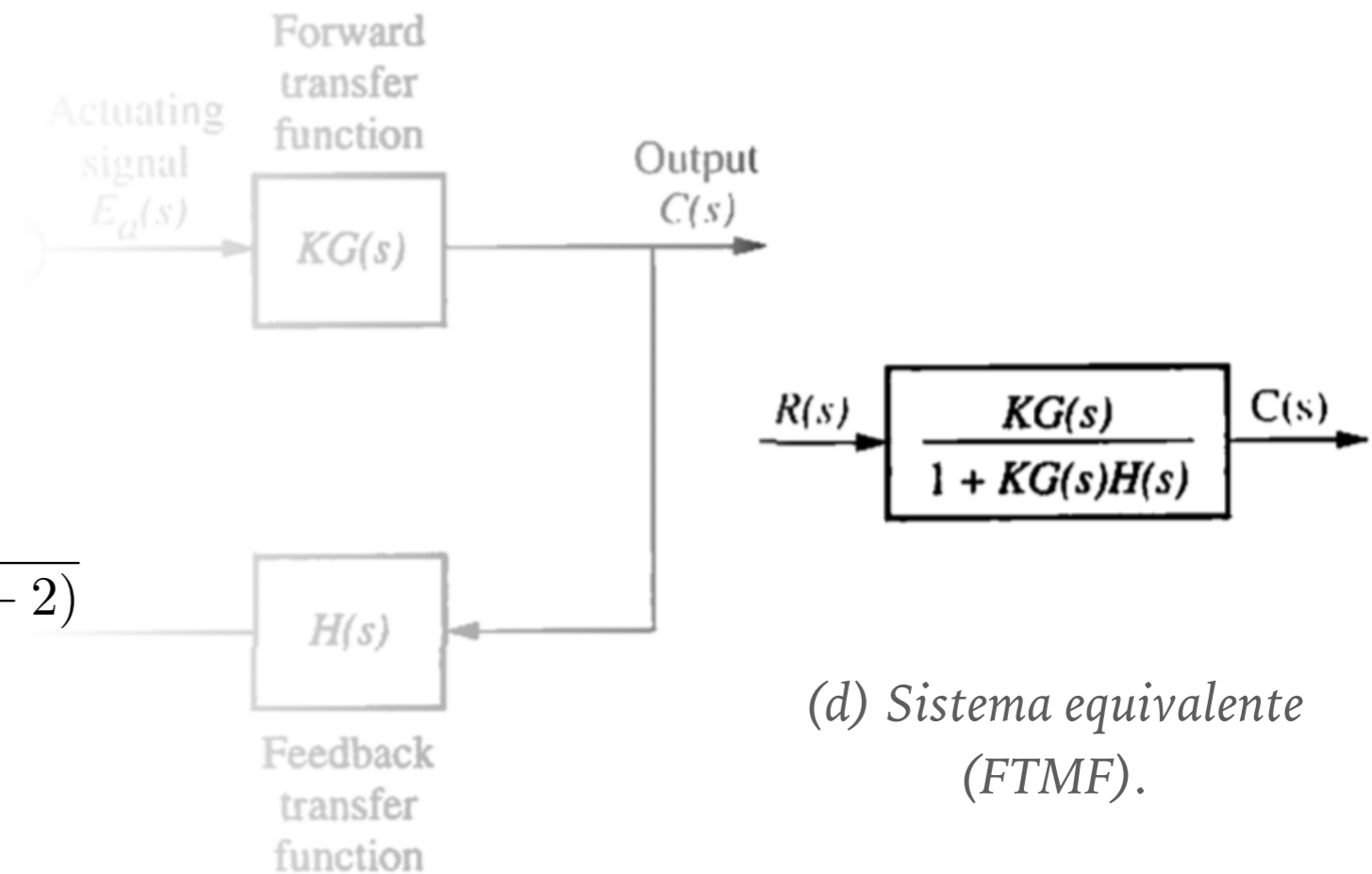
(c) Fechando a malha:

$$FTMF(s) = \frac{K G(s)}{1 + K G(s)} = \frac{R(s)}{C(s)}$$

$$FTMF(s) = \frac{\frac{K(s+2)}{(s+5)}}{1 + \frac{K(s+2)}{(s+5)}} = \frac{K(s+2)}{(s+5) + K(s+2)}$$

$$FTMF(s) = \frac{K(s+2)}{(K+1)s + (2K+5)}$$

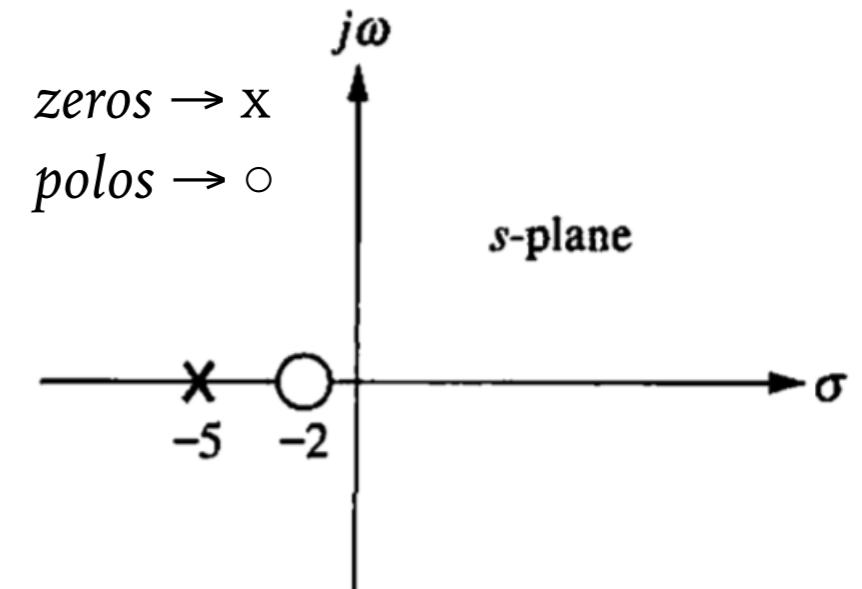
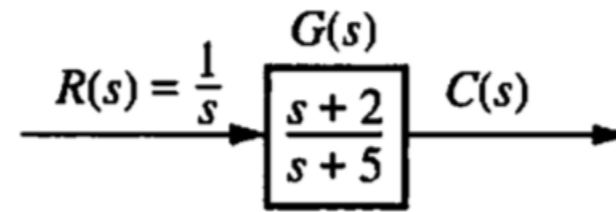
$EC(s)$



(d) Sistema equivalente (FTMF).

# POLOS E ZEROS DE UM SISTEMA

- Seja o seguinte sistema:
- O que acontece quando fechamos a malha com controlador proporcional?



(a) Sistema em MA. (b) Pólos e zeros de MA no plano-s

(c) Fechando a malha:

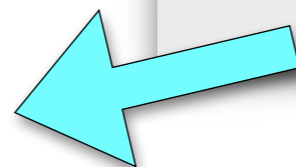
$$FTMF(s) = \frac{K G(s)}{1 + K G(s)} = \frac{R(s)}{C(s)}$$

$$FTMF(s) = \frac{K(s+2)}{(K+1)s + (2K+5)}$$

Calculando pólos de MF, variando K:

K	EC(s)=0	Polo em (s=)
0.0	1 s + 5 = 0	-5.00
0.1	1.1 s + 5.2 = 0	-4.73
0.5	1.5 s + 6 = 0	-4.00
1.0	2 s + 7 = 0	-3.50
1.5	2.5 s + 8 = 0	-3.20
2.0	3 s + 9 = 0	-3.00
4.0	5 s + 13 = 0	-2.60
10.0	11 s + 25 = 0	-2.27
50.0	51 s + 105 = 0	-2.06
100.0	101 s + 205 = 0	-2.03

```
% Determinando faixa de polos em MF,
% variando ganho para fig. 4.1 NISE
% Fernando Passold, em 01.04.2019
K=[0 0.1 0.5 1 1.5 2 4 10 50 100];
u=length(K);
fprintf(' K | EC(s)=0 | Polo em (s=)\n');
for i=1:u
    EC = [(K(i) + 1) (2*K(i) + 5)]; % monta EC(s)
    polo = roots(EC);
    fprintf('%5.1f | %g s + %g = 0 | %7.2f\n', K(i), EC(1),
    EC(2), polo);
end
```

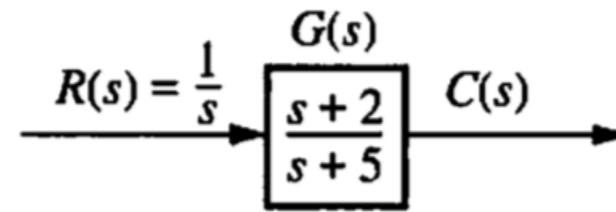


Feedback transfer function

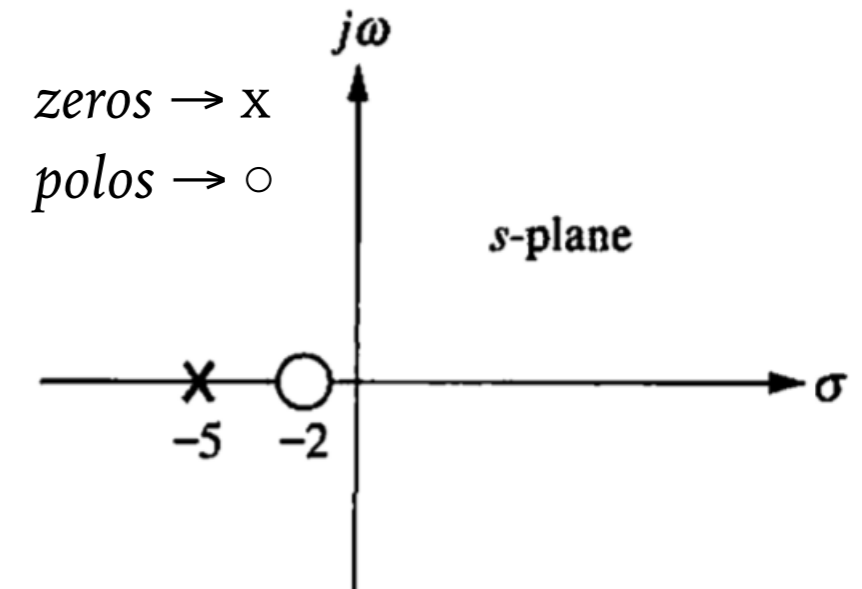
(FTMF).

# POLOS E ZEROS DE UM SISTEMA

- Seja o seguinte sistema:
- O que acontece quando fechamos a malha com controlador proporcional?



(a) Sistema em MA.



(b) Pólos e zeros de MA no plano-s

(c) Fechando a malha:

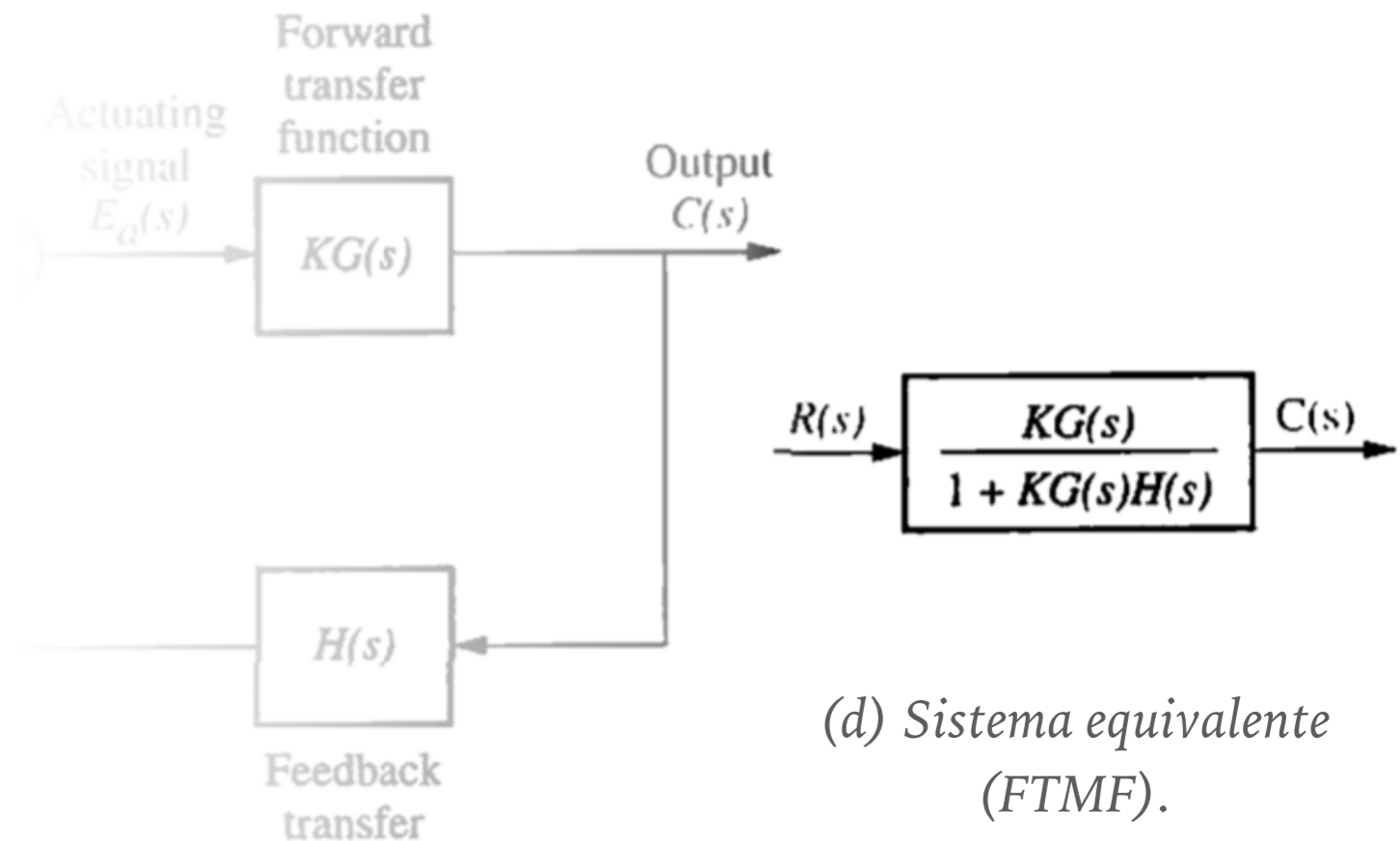
$$FTMF(s) = \frac{K G(s)}{1 + K G(s)} = \frac{R(s)}{C(s)}$$

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1.5	2.5 s + 8 = 0	-3.20
2.0	3 s + 9 = 0	-3.00
4.0	5 s + 13 = 0	-2.60
10.0	11 s + 25 = 0	-2.27
50.0	51 s + 105 = 0	-2.06
100.0	101 s + 205 = 0	-2.03

Note: o pólo de MA "caminha" na direção do zero (mais próximo)

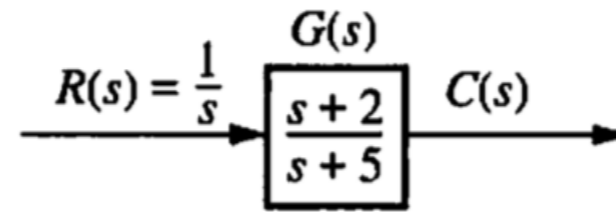


(d) Sistema equivalente (FTMF).

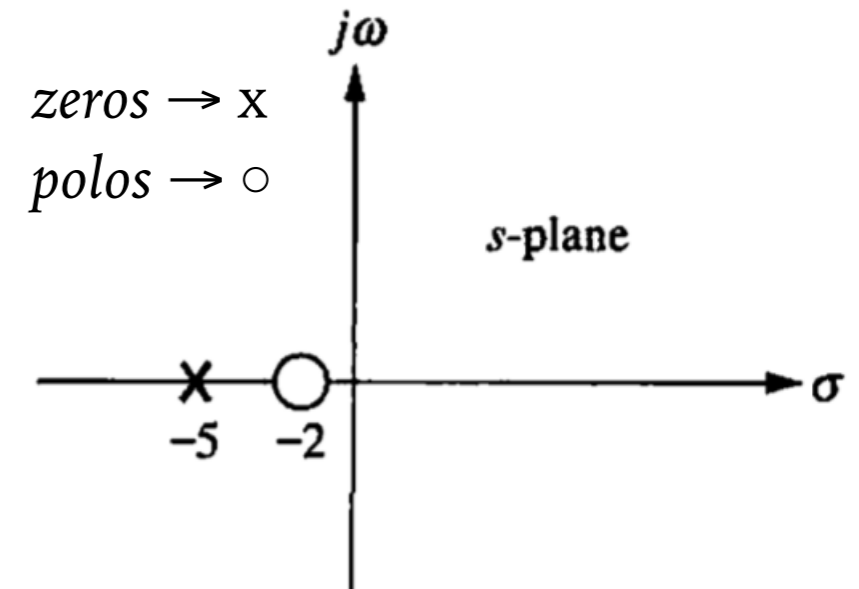


# POLOS E ZEROS DE UM SISTEMA

```
>> G=tf([1 2],[1 5])
G =
    s + 2
    ----
    s + 5
>> rlocus(G)
```



(a) Sistema em MA.



(b) Pólos e zeros de MA no plano-s

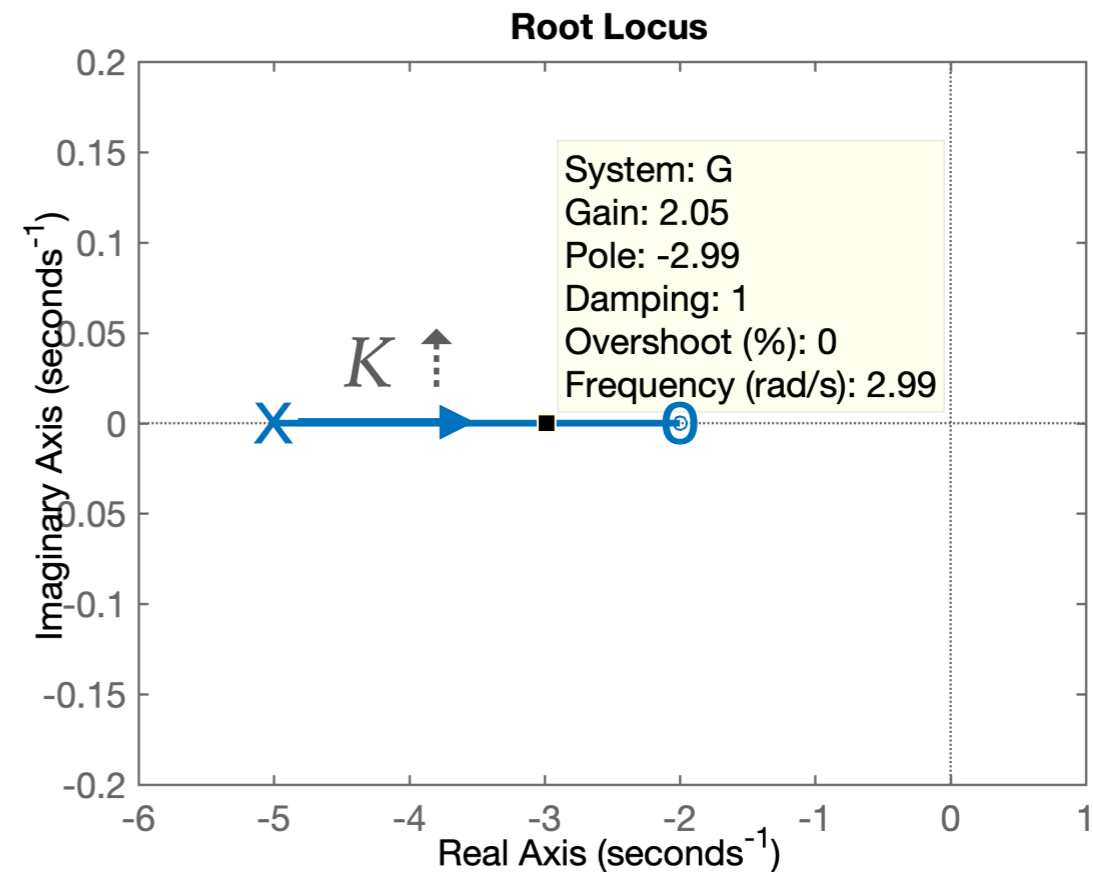
(c) Fechando a malha:

$$FTMF(s) = \frac{K G(s)}{1 + K G(s)} = \frac{R(s)}{C(s)}$$

$$FTMF(s) = \frac{K(s + 2)}{(K + 1)s + (2K + 5)}$$

Calculando pólos de MF, variando K:

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4.0	5 s + 13 = 0	-2.60
10.0	11 s + 25 = 0	-2.27
50.0	51 s + 105 = 0	-2.06
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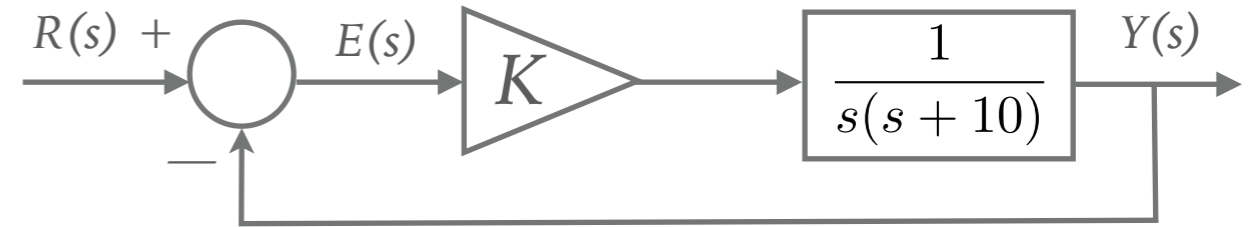
Note: o pólo de MA “caminha” na direção do zero (mais próximo)



# EX\_2: SISTEMA DE 2ª-ORDEM (SOMENTE 2 POLOS)

► Seja o seguinte sistema:

*Pólos de MA em  $s=0$  e  $s=-10$ .*



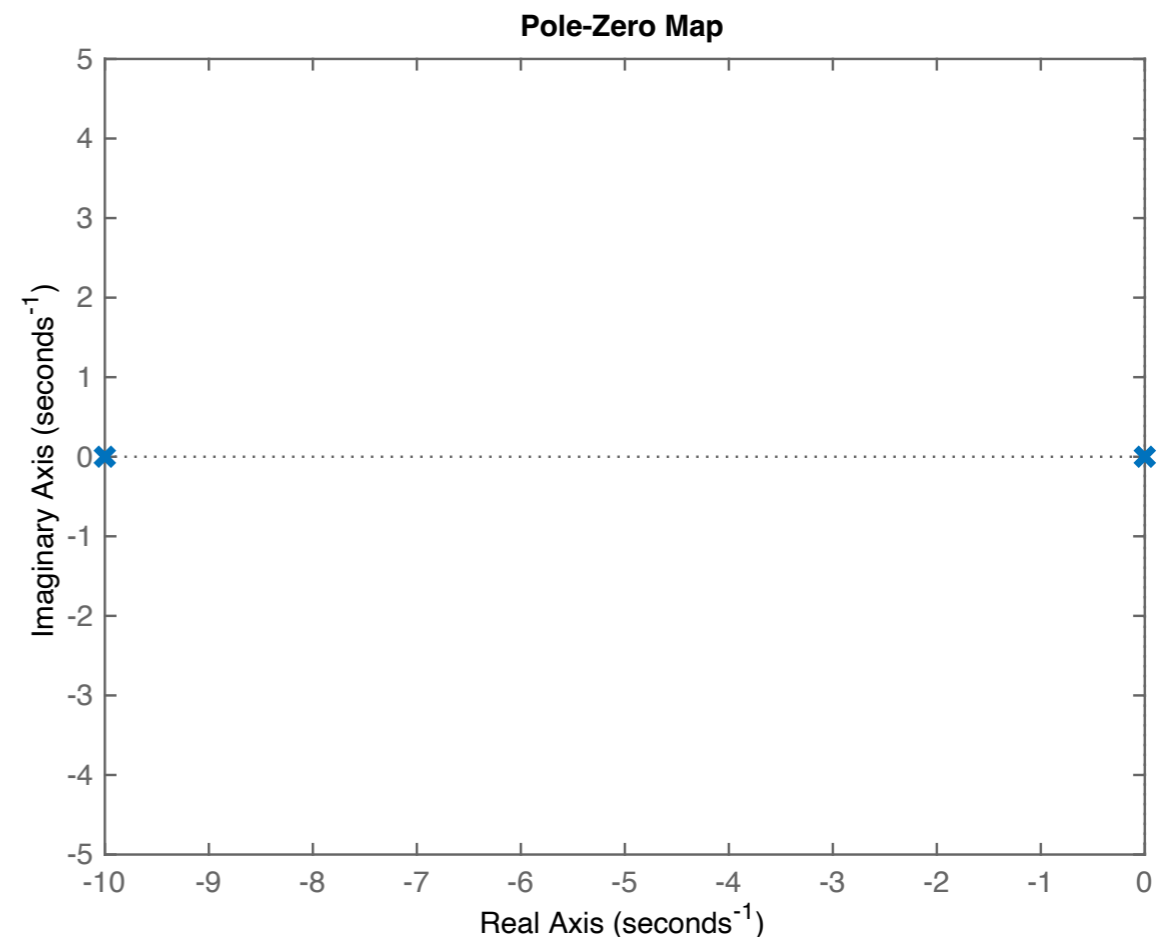
$$FTMF(s) = \frac{K G(s)}{1 + K G(s)} = \frac{Y(s)}{R(s)}$$

$$FTMF(s) = \frac{\frac{K}{s(s+10)}}{1 + \frac{K}{s(s+10)}} = \frac{K}{s(s+10) + K}$$

$$FTMF(s) = \frac{K}{s^2 + 10s + K}$$

*Variando K obteremos os pólos de MF em:*

K	Polo 1	Polo 2
0	0	-10
5	-9.47214	-0.527864
10	-8.87298	-1.12702
15	-8.16228	-1.83772
20	-7.23607	-2.76393
25	-5	-5
30	$-5 + j2.23607$	$-5 - j2.23607$
35	$-5 + j3.16228$	$-5 - j3.16228$
40	$-5 + j3.87298$	$-5 - j3.87298$
45	$-5 + j4.47214$	$-5 - j4.47214$
50	$-5 + j5$	$-5 - j5$



```
>> G=tf(1,[1 10 0])
>> pzmap(G)
>> axis([-10 0 -5 5])
```

# EX\_2: SISTEMA DE 2ª-ORDEM (COMENTE O POLO)

► Seja o seguinte sistema:

Pólos de MA em  $s=0$  e  $s=-10$ .

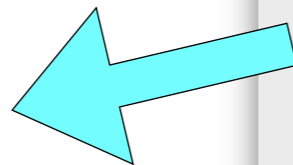
$$FTMF(s) = \frac{K G(s)}{1 + K G(s)} = \frac{Y(s)}{R(s)}$$

$$FTMF(s) = \frac{\frac{K}{s(s+10)}}{1 + \frac{K}{s(s+10)}} = \frac{K}{s(s+10)}$$

$$FTMF(s) = \frac{K}{s^2 + 10s + K}$$

Variando  $K$  obteremos os pólos de MF e

$K$	Polo 1	Polo 2
0	0	-10
5	-9.47214	-0.527864
10	-8.87298	-1.12702
15	-8.16228	-1.83772
20	-7.23607	-2.76393
25	-5	-5
30	$-5 + j2.23607$	$-5 - j2.23607$
35	$-5 + j3.16228$	$-5 - j3.16228$
40	$-5 + j3.87298$	$-5 - j3.87298$
45	$-5 + j4.47214$	$-5 - j4.47214$
50	$-5 + j5$	$-5 - j5$

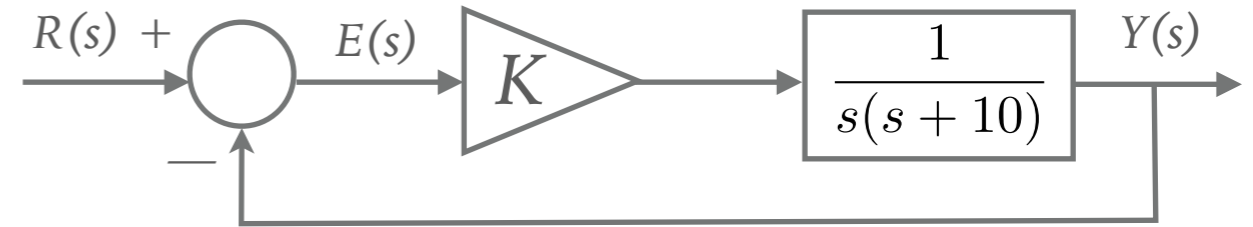


```
% Determinando faixa de p?los em MF, variando ganho para fig. 8.4 NISE
% Fernando Passold, em 01.04.2019
K=0:5:50;
u=length(K);
fprintf(' K & \\text{Polo 1} & \\text{Polo 2} \\\\ \\n');
figure;
for i=1:u
    fprintf('%2g & ', K(i));
    EC = [1 10 K(i)]; % monta EC(s) (e mostra polin?mio)
    polo = roots(EC);
    fprintf('%g ', real(polo(1)));
    aux=num2str(K(i));
    if ~(isreal(polo(1)))
        plot(real(polo),imag(polo),'bx','LineWidth',2,'MarkerSize',12)
        text(real(polo)+.2,imag(polo),aux);
        aux=abs(imag(polo(j)));
        fprintf('+ j%g ', aux);
    else
        plot(real(polo),[0 0],'bx','LineWidth',2,'MarkerSize',12)
        text(real(polo)+.2,[0.2 0.2],aux);
    end
    fprintf(' & %g ', real(polo(2)));
    if ~(isreal(polo(1)))
        fprintf('- j%g ', aux);
    end
    if i==1
        hold on
    end
    fprintf(' \\\\ \\n');
end
title('Plano-s');
xlabel('Real (\\sigma)');
ylabel('Imag (j\\omega)');
```

# EX\_2: SISTEMA DE 2ª-ORDEM (SOMENTE 2 POLOS)

► Seja o seguinte sistema:

Pólos de MA em  $s=0$  e  $s=-10$ .



$$FTMF(s) = \frac{K G(s)}{1 + K G(s)} = \frac{Y(s)}{R(s)}$$

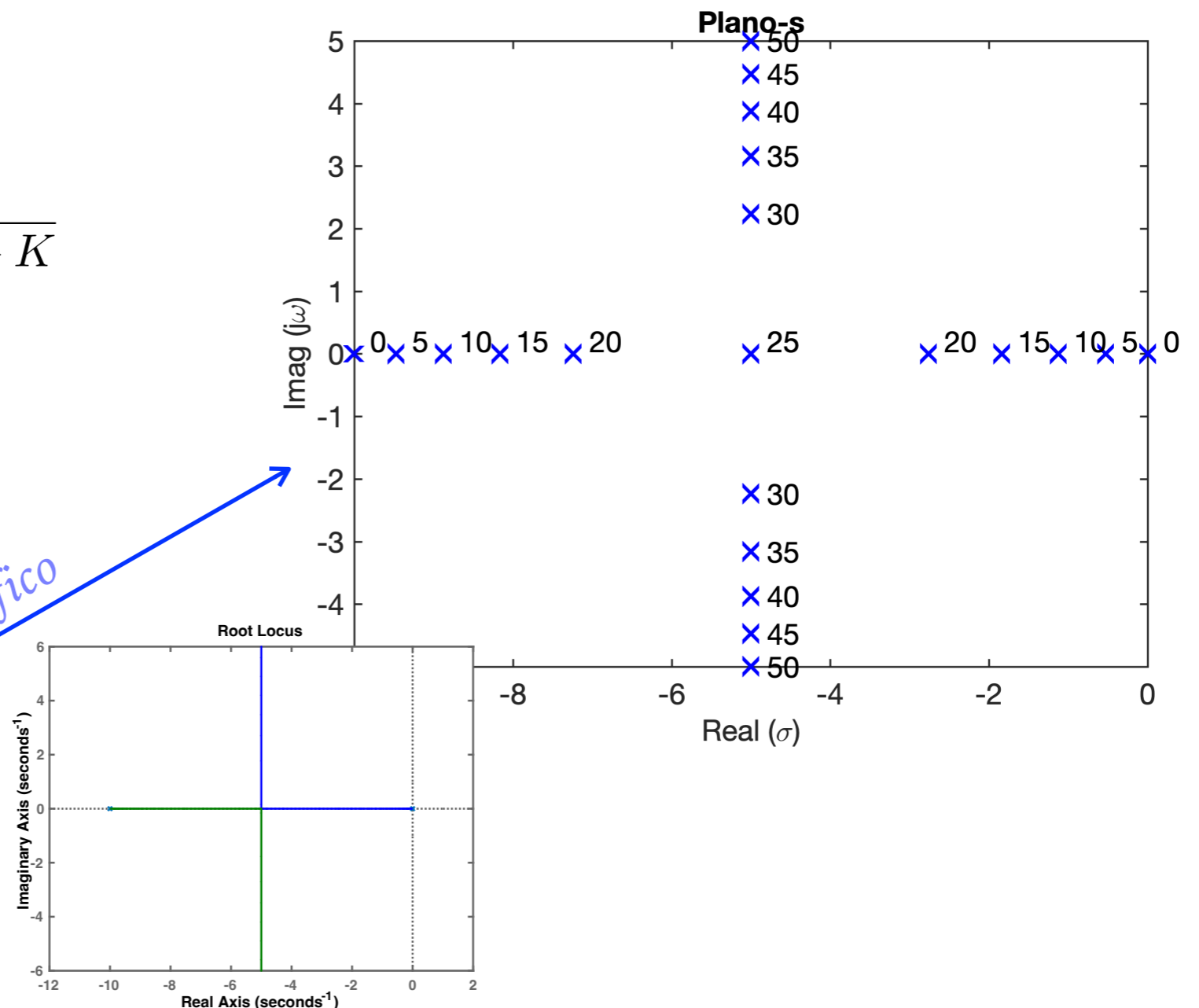
$$FTMF(s) = \frac{\frac{K}{s(s+10)}}{1 + \frac{K}{s(s+10)}} = \frac{K}{s(s+10) + K}$$

$$FTMF(s) = \frac{K}{s^2 + 10s + K}$$

Variando  $K$  obteremos os pólos de MF em:

$K$	Polo 1	Polo 2
0	0	-10
5	-9.47214	-0.527864
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45	$-5 + j4.47214$	$-5 - j4.47214$
50	$-5 + j5$	$-5 - j5$

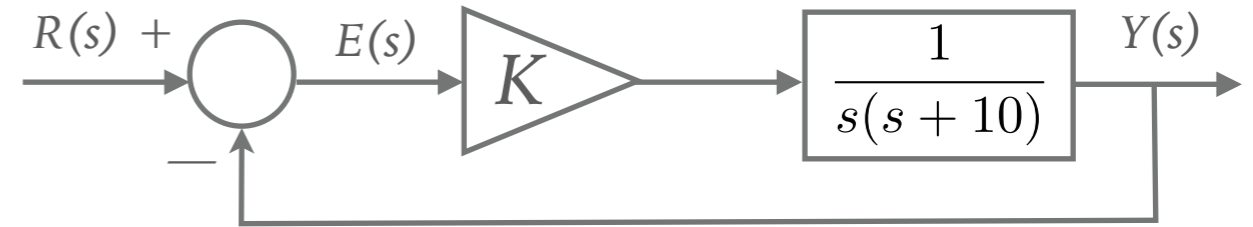
Gráfico



# EX\_2: SISTEMA DE 2ª-ORDEM (SOMENTE 2 POLOS)

► Seja o seguinte sistema:

Pólos de MA em  $s=0$  e  $s=-10$ .



$$FTMF(s) = \frac{K G(s)}{1 + K G(s)} = \frac{Y(s)}{R(s)}$$

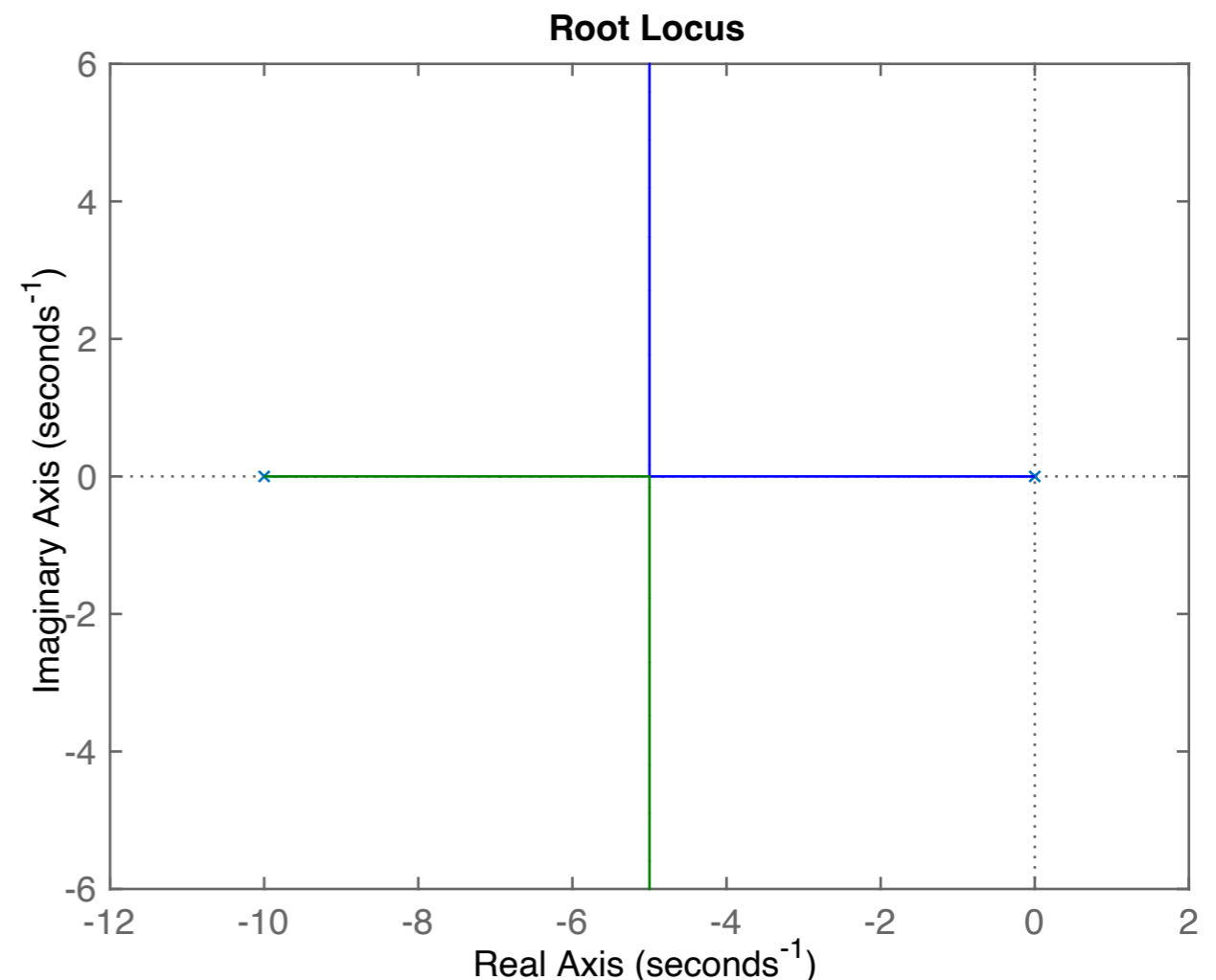
$$FTMF(s) = \frac{\frac{K}{s(s+10)}}{1 + \frac{K}{s(s+10)}} = \frac{K}{s(s+10) + K}$$

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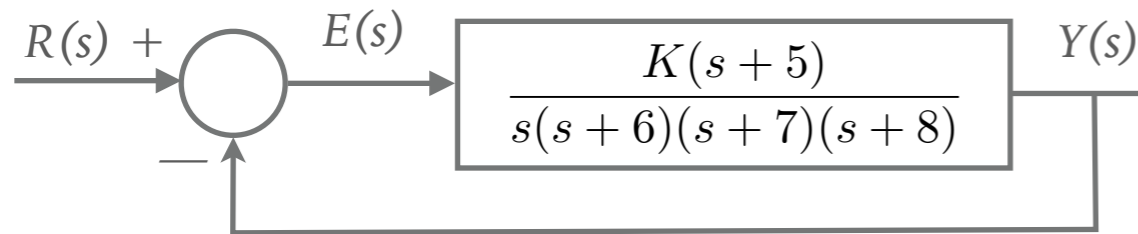
Gráfico



```
>> figure; rlocus(G)
```

# EX\_3: VOLTANDO AO EXEMPLO DO INÍCIO DA AULA

► Seja este outro sistema:



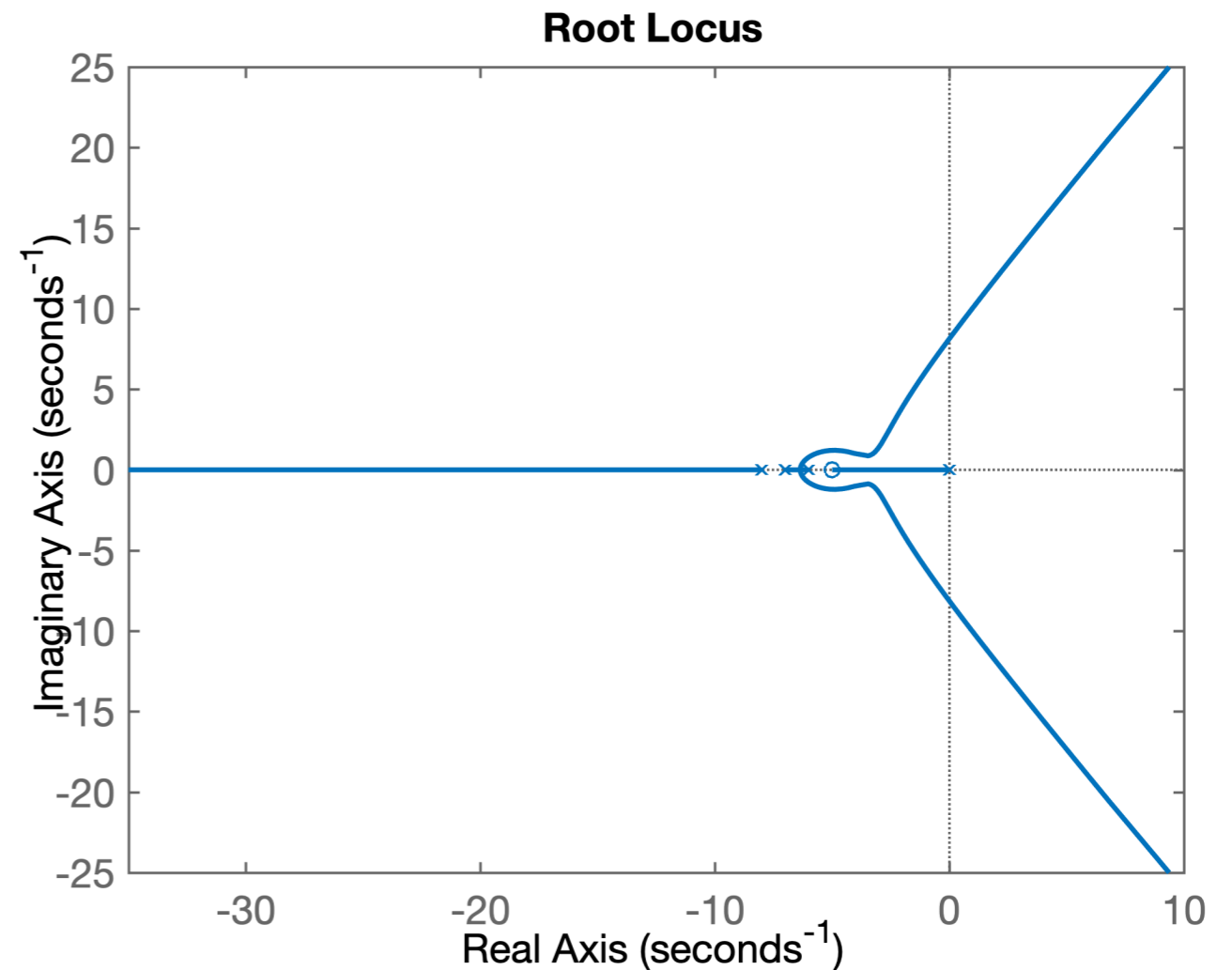
```
>> G=tf([1 5],poly([0 -6 -7 -8]));  
>> zpk(G)
```

ans =

$$\frac{(s+5)}{s(s+8)(s+7)(s+6)}$$

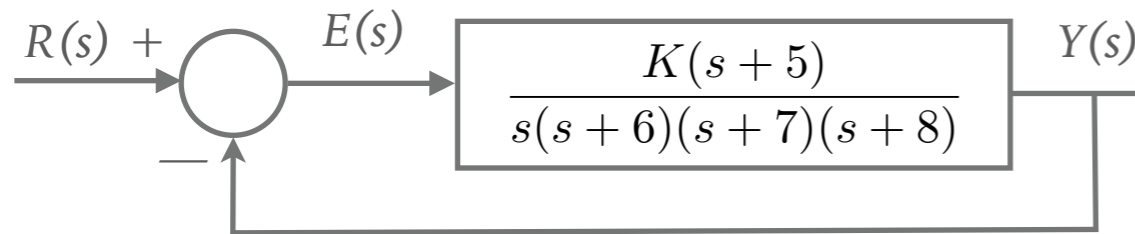
Continuous-time zero/pole/gain model.

```
>> rlocus(G)
```



# EX\_3: VOLTANDO AO EXEMPLO DO INÍCIO DA AULA

► Seja este outro sistema:



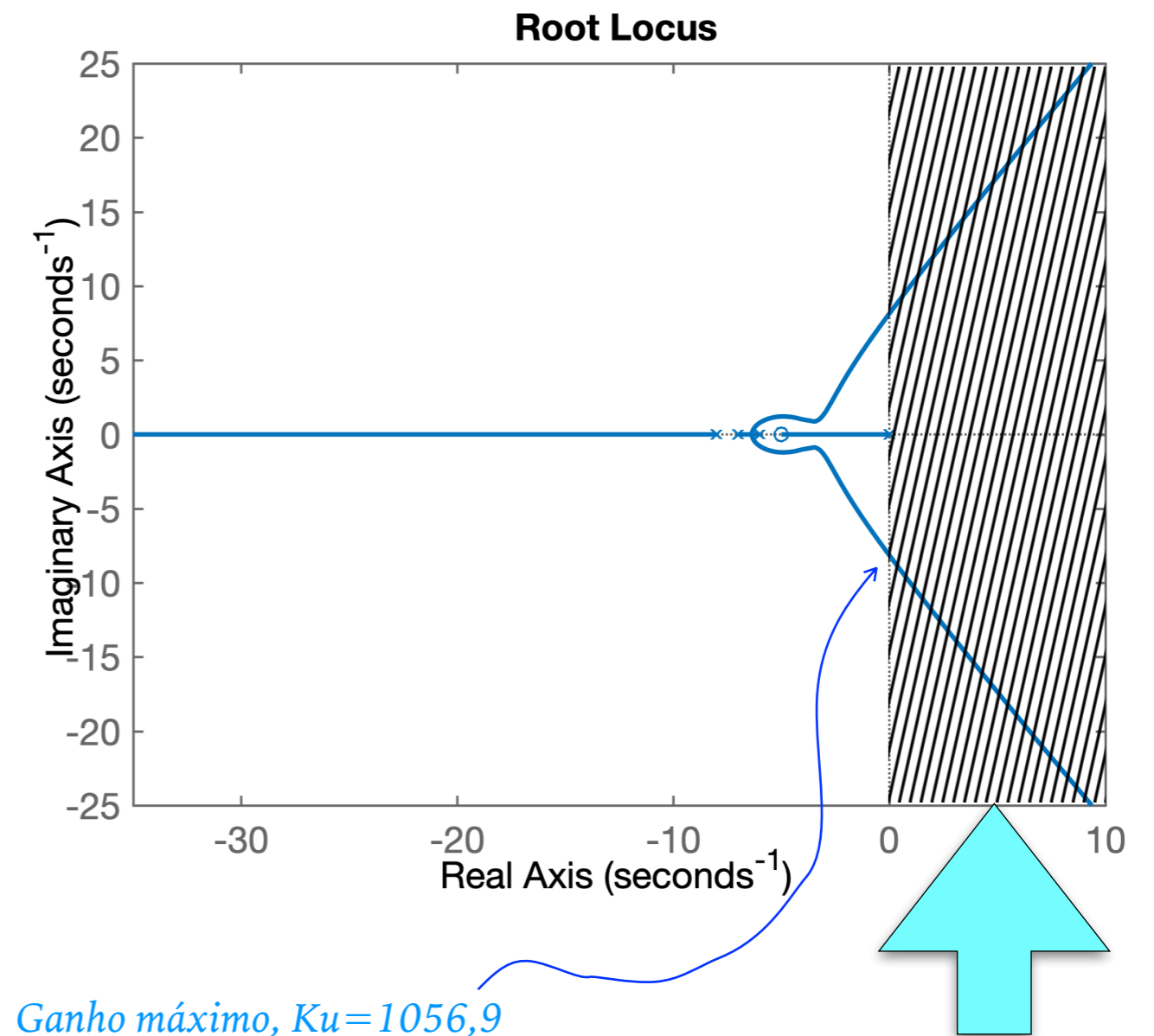
```
>> G=tf([1 5],poly([0 -6 -7 -8]));  
>> zpk(G)
```

ans =

$$\frac{(s+5)}{s(s+8)(s+7)(s+6)}$$

Continuous-time zero/pole/gain model.

```
>> rlocus(G)
```



Região de  
instabilidade

# EX\_3: VOLTANDO AO EXEMPLO DO INÍCIO DA AULA

► Seja o sistema:

```
>> G=tf([1 5],poly([0 -6 -7 -8]));
>> zpk(G)
```

$$\frac{(s+5)}{s(s+8)(s+7)(s+6)}$$

```
>> rlocus(G)
```

```
>> G
```

```
G =
```

$$\frac{s + 5}{s^4 + 21s^3 + 146s^2 + 336s}$$

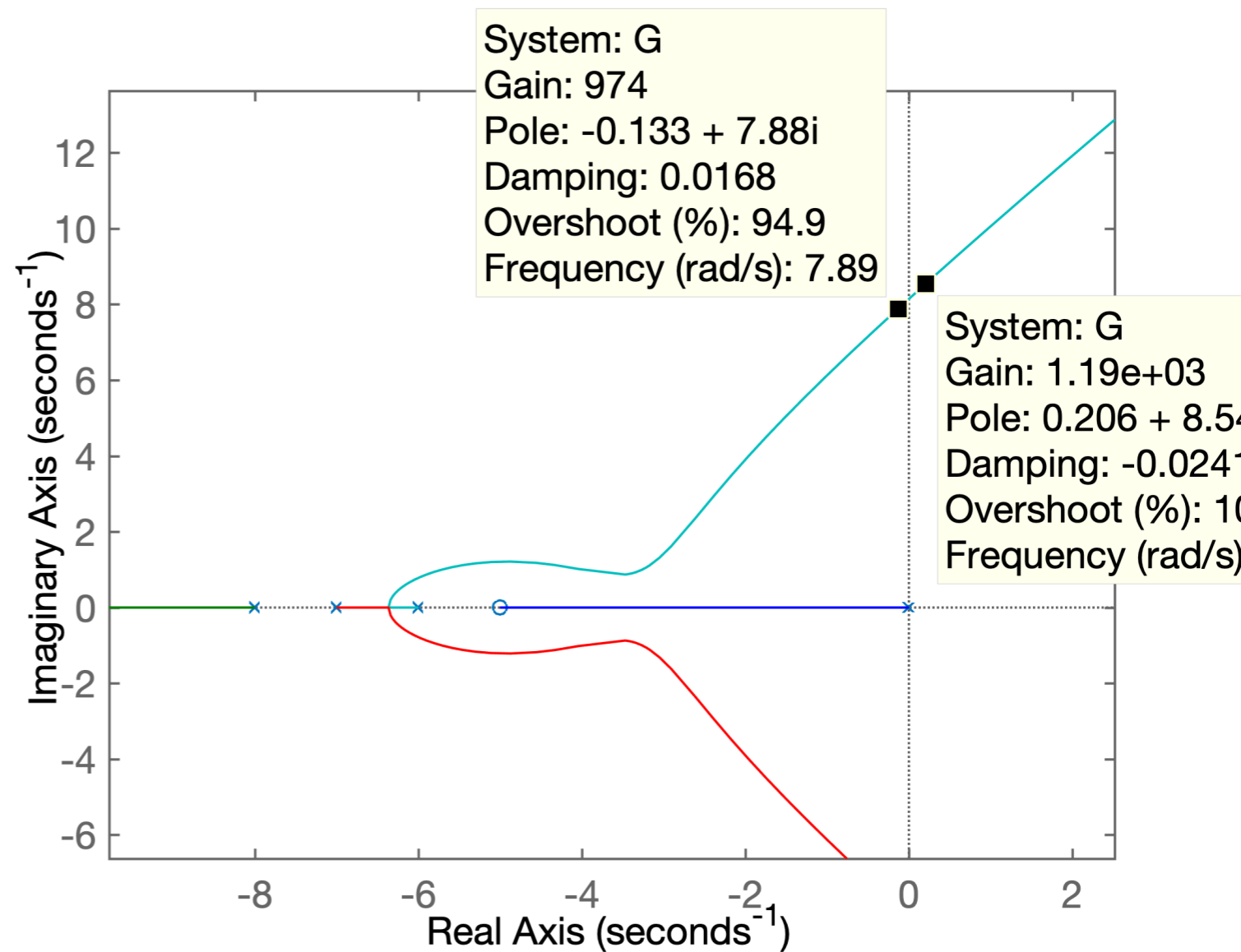
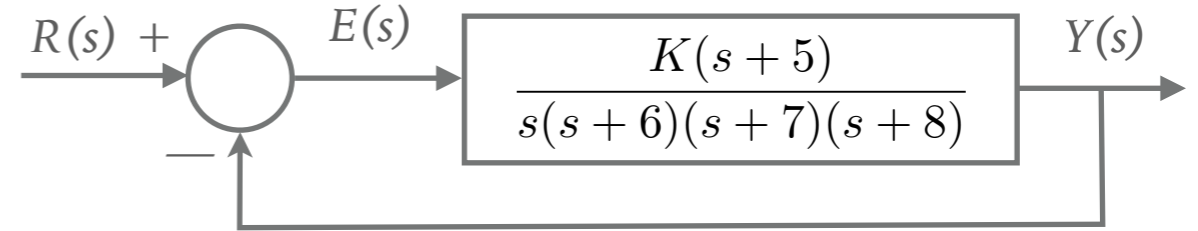
Ganho máximo,  $K_u = 1056,9$

$$FTMF(s) = \frac{K(s+5)}{s^4 + 21s^3 + 146s^2 + 336s + K(s+5)}$$

$$EC(z) = s^4 + 21s^3 + 146s^2 + (K + 336)s + 5K = 0$$

$s^4$	1	146	$5K$
$s^3$	21	$(K + 336)$	0
$s^2$	$b_1$	$b_2$	
$s^1$	$c_1$		
$s^0$	$d_1$		

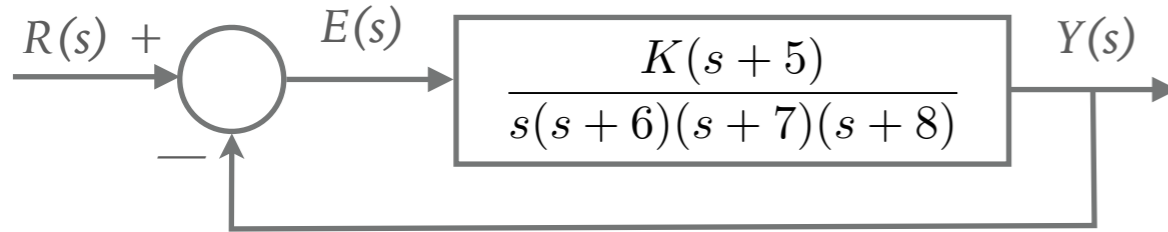
Arranjo de Routh-Hurwitz





# EX\_3: VOLTANDO AO EXEMPLO DO INÍCIO DA AULA

► Seja o sistema:



Note:

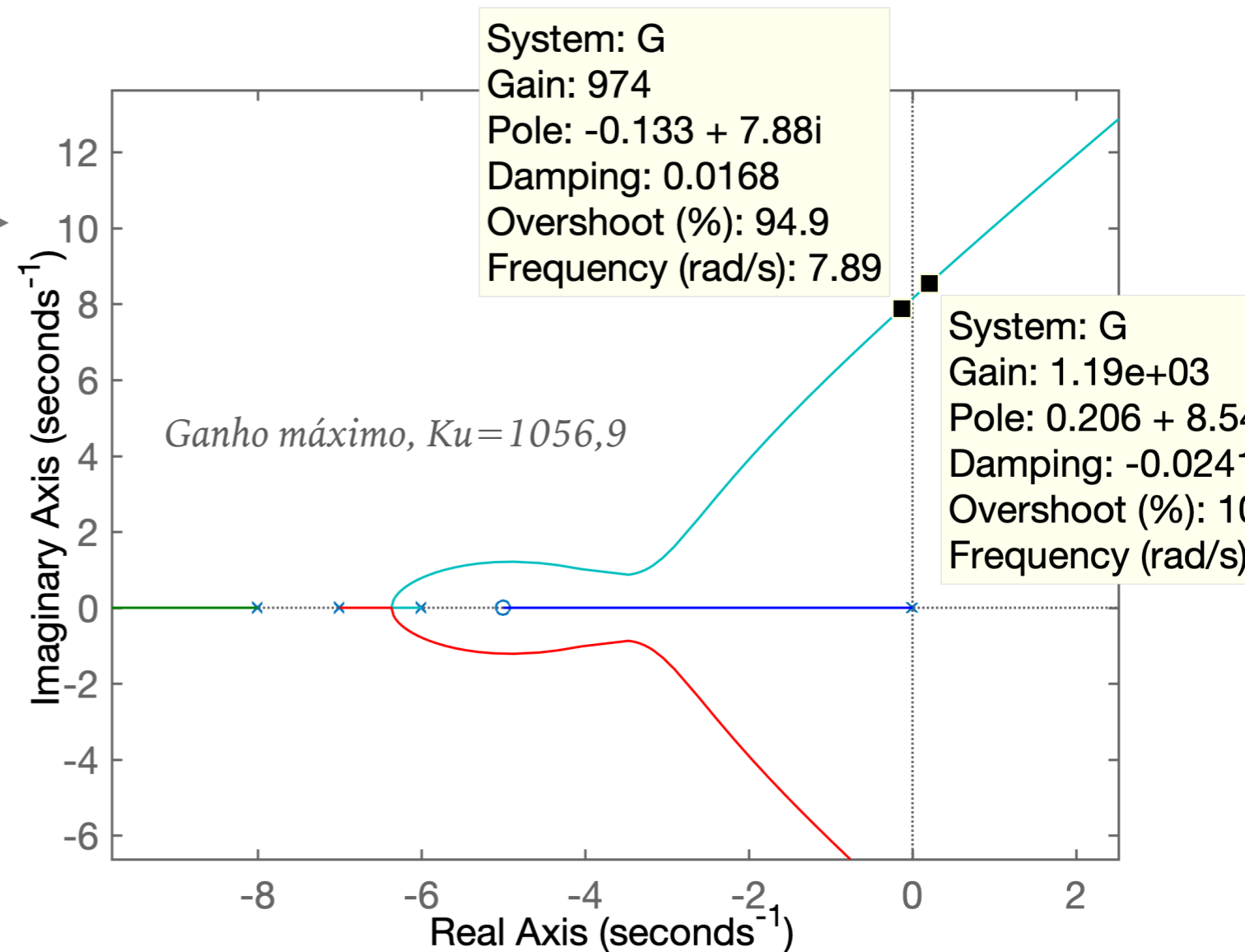
Se  $K=1$ , teremos...

Se  $K=10$ , teremos...

Se  $K=100$ , teremos...

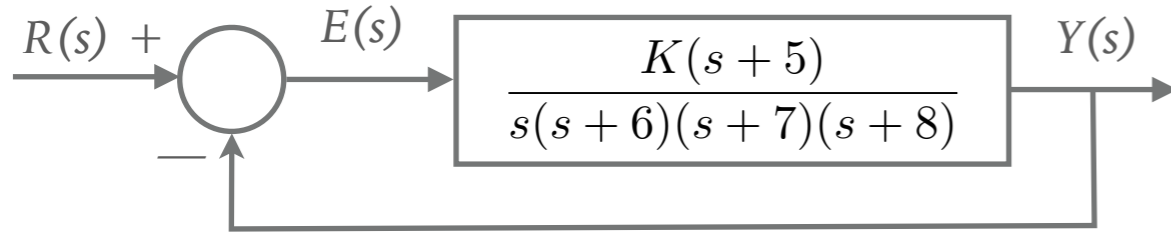
Se  $K=1000$ , teremos...

Se  $K=1100$ , teremos...



# EX\_3: VOLTANDO AO EXEMPLO DO INÍCIO DA AULA

► Seja o sistema:



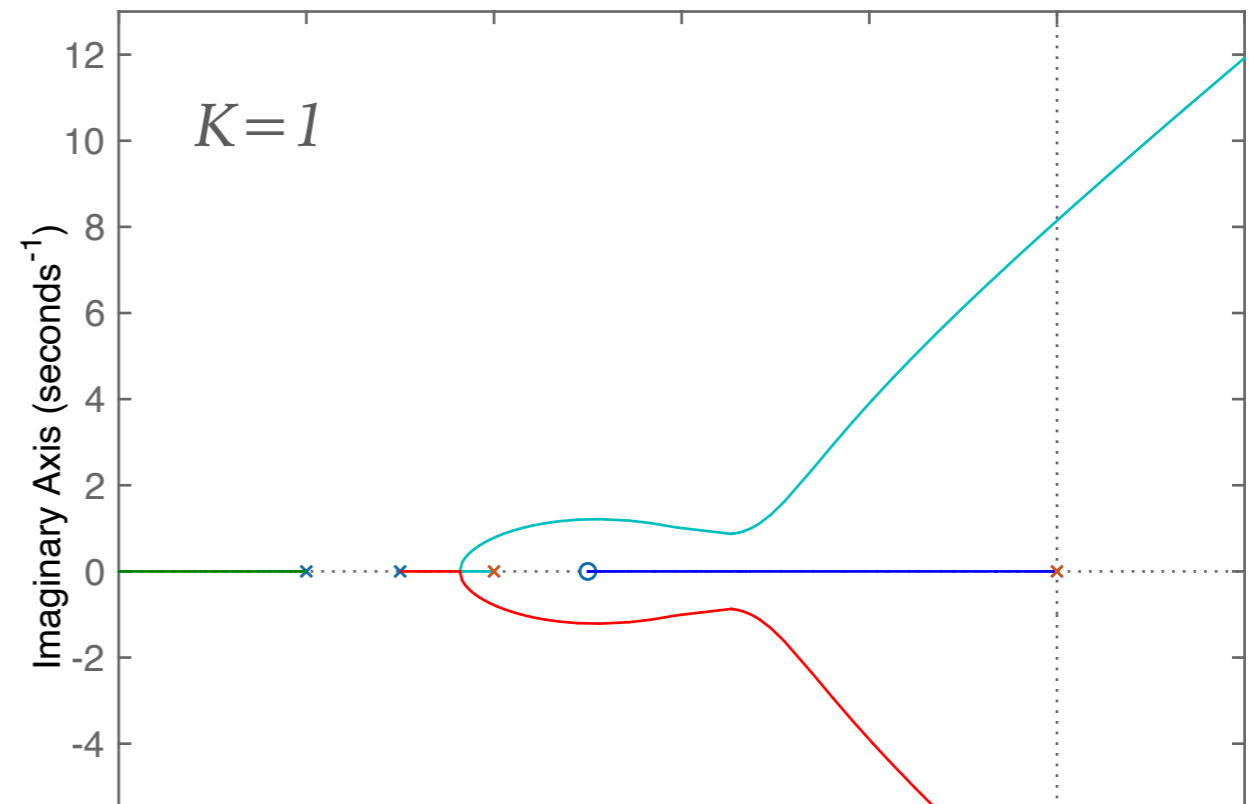
Note:

Ganho máximo,  $K_u = 1056,9$

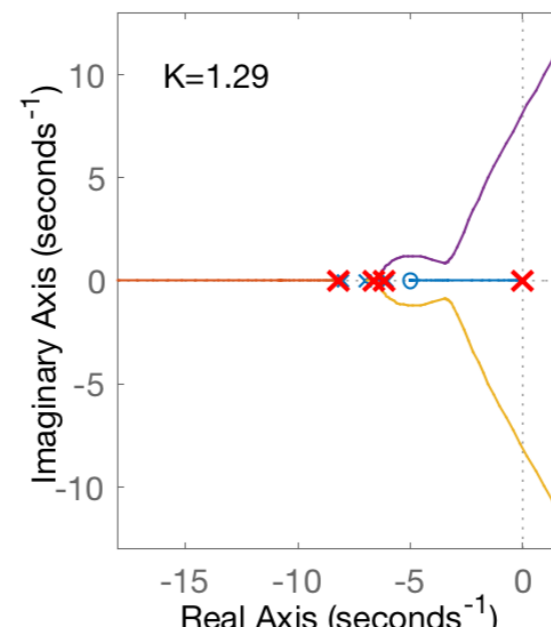
Se  $K=1$ , teremos...

```
>> K=1;
>> FTMF=feedback(K*G,1);
>> pole(FTMF)
ans =
-8.1554
-6.7224
-6.1073
-0.014933
>> zpk(FTMF)
(s+5)
-----
(s+8.155) (s+6.722) (s+6.107) (s+0.01493)
>>
```

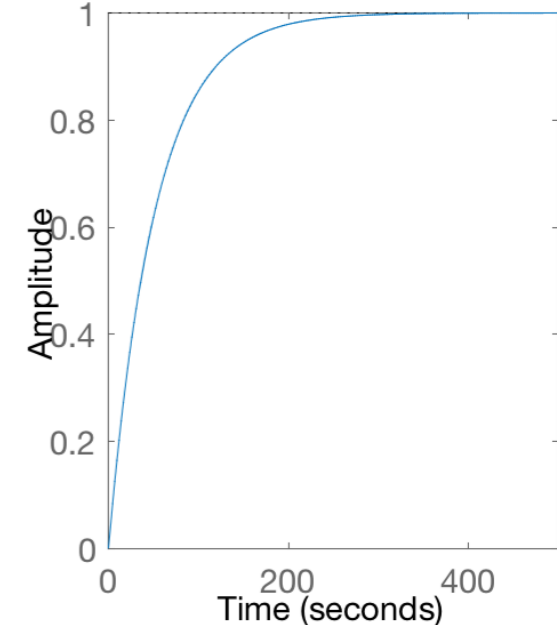
Root Locus



Pole-Zero Map

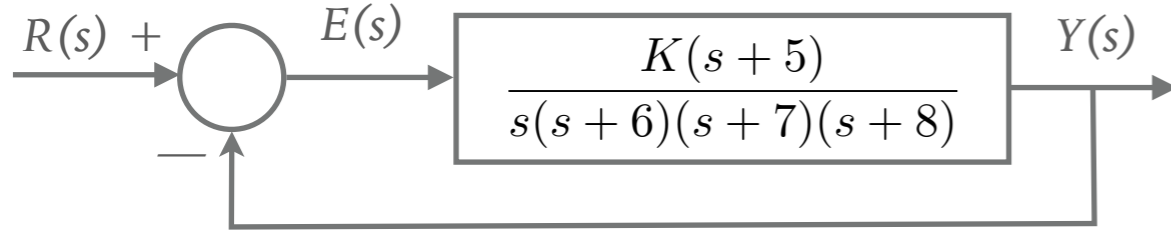


Step Response



# EX\_3: VOLTANDO AO EXEMPLO DO INÍCIO DA AULA

► Seja o sistema:



Note:

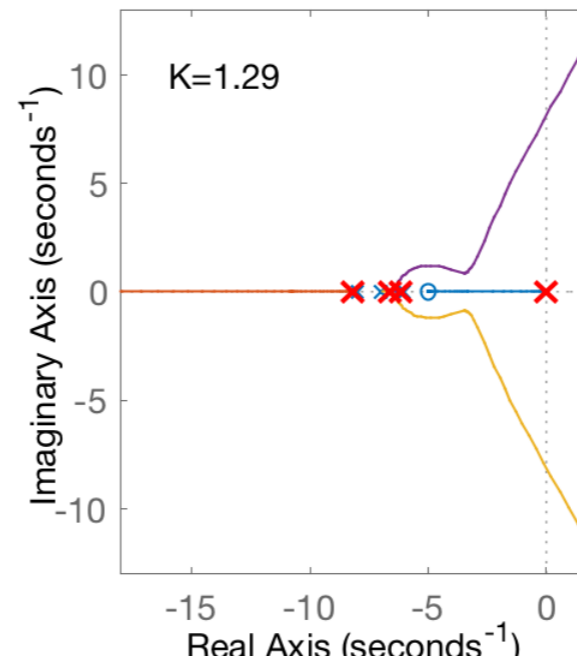
Ganho máximo,  $K_u = 1056,9$

Se  $K=1$ , teremos...

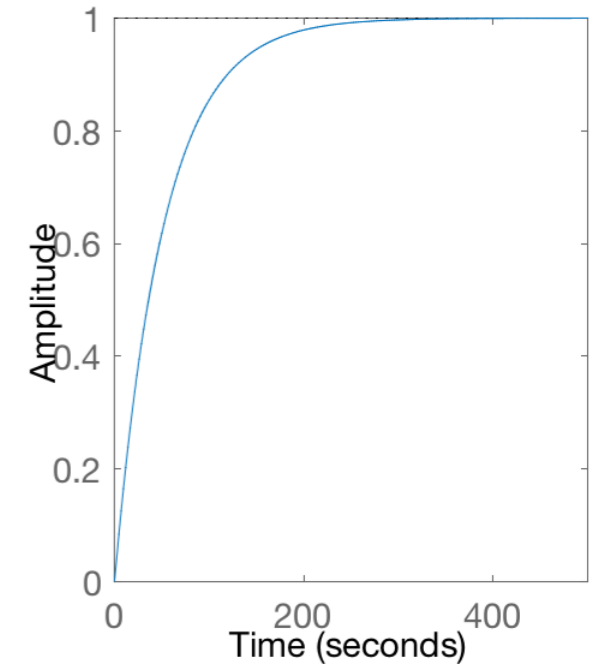
Se  $K=10$ , teremos...

```
>> K=10;
>> FTMF=feedback(K*G,1);
>> pole(FTMF)
ans =
-8.8346 + 0i
-6.0055 + 0.77683i
-6.0055 - 0.77683i
-0.15434 + 0i
>> zpk(FTMF)
10 (s+5)
-----
(s+8.835) (s+0.1543) (s^2 + 12.01s + 36.67)
>>
```

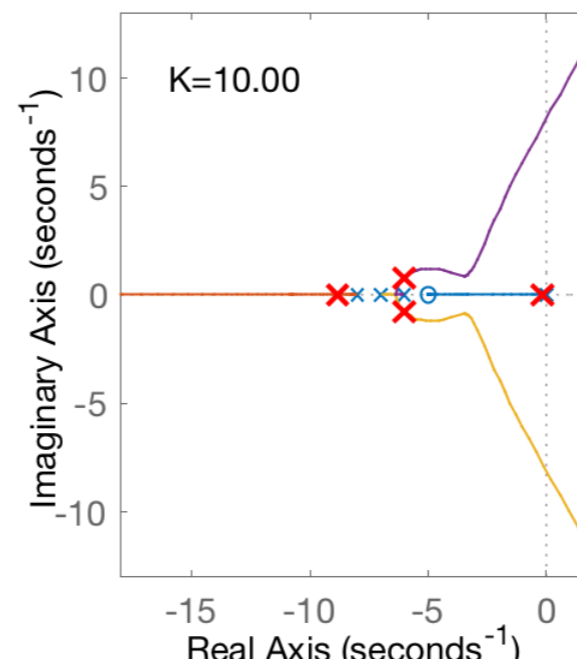
Pole-Zero Map



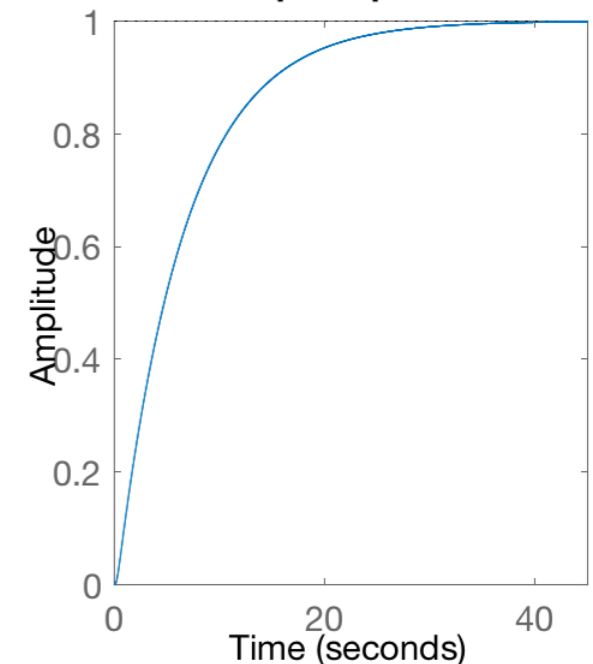
Step Response



Pole-Zero Map

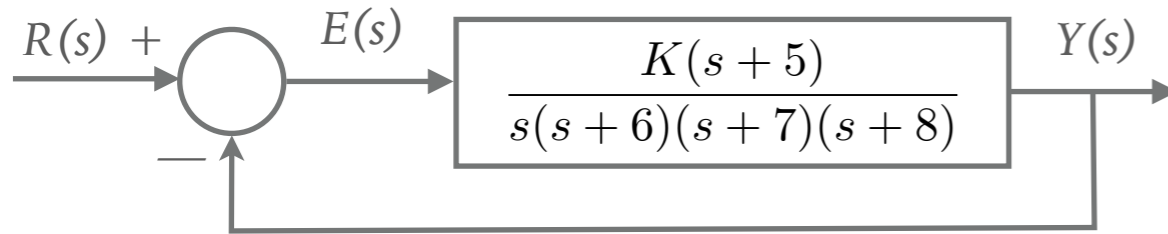


Step Response



# EX\_3: VOLTANDO AO EXEMPLO DO INÍCIO DA AULA

► Seja o sistema:



Note:

Ganho máximo,  $K_u = 1056,9$

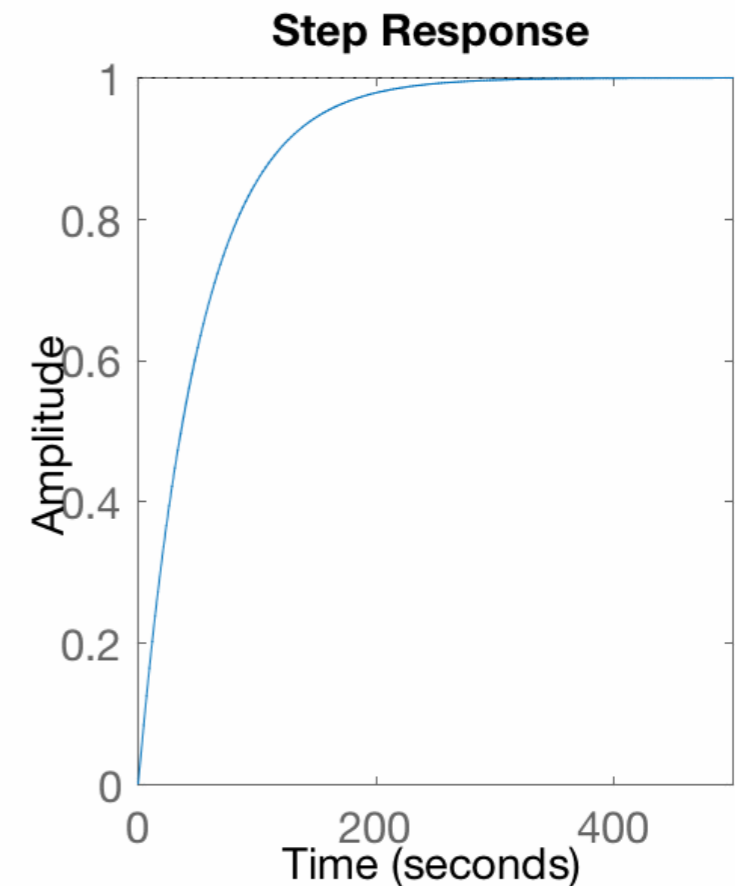
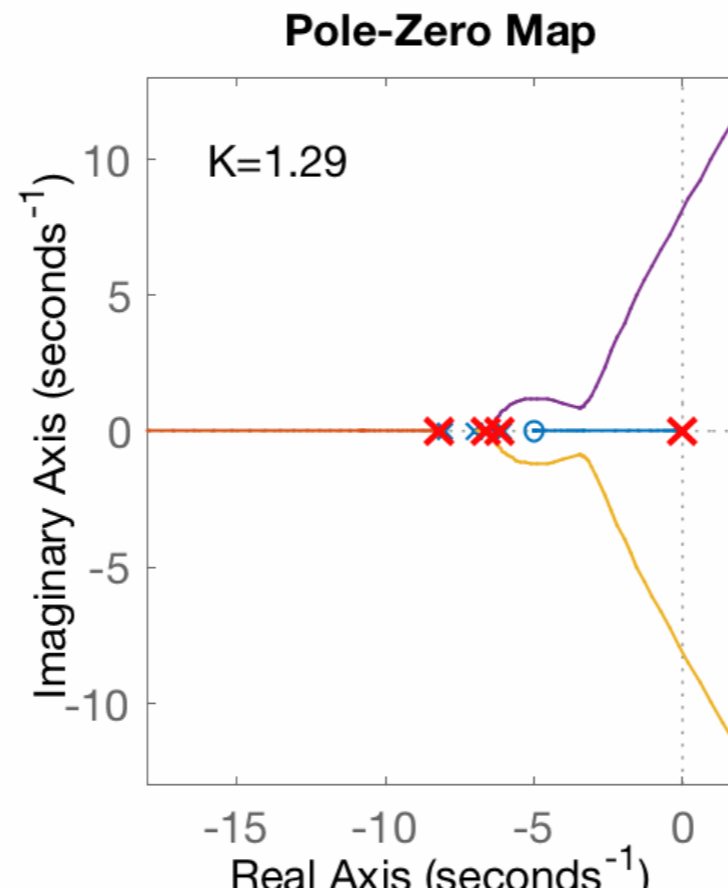
Se  $K=1$ , teremos...

Se  $K=10$ , teremos...

Se  $K=100$ , teremos...

Se  $K=1000$ , teremos...

Se  $K=1100$ , teremos...



# PROPRIEDADES (REGRAS) DO ROOT LOCUS (RL)

$$FTMF(s) = \frac{K \cdot G(s)}{1 + K \cdot G(s)H(s)}$$

$$EC(z) = 1 + K \cdot G(s)H(s) = 0$$

$$K \cdot G(s)H(s) = -1 = 1 \angle [(2k + 1) \cdot 180^\circ], \quad \text{onde: } k = 0, \pm 1, \pm 2, \pm 3, \dots$$

$$|K \cdot G(s)H(s)| = 1$$

$$\angle KG(s)H(s) = (2k + 1) \cdot 180^\circ$$

Para um ponto no plano-s pertencer ao traço do RL:

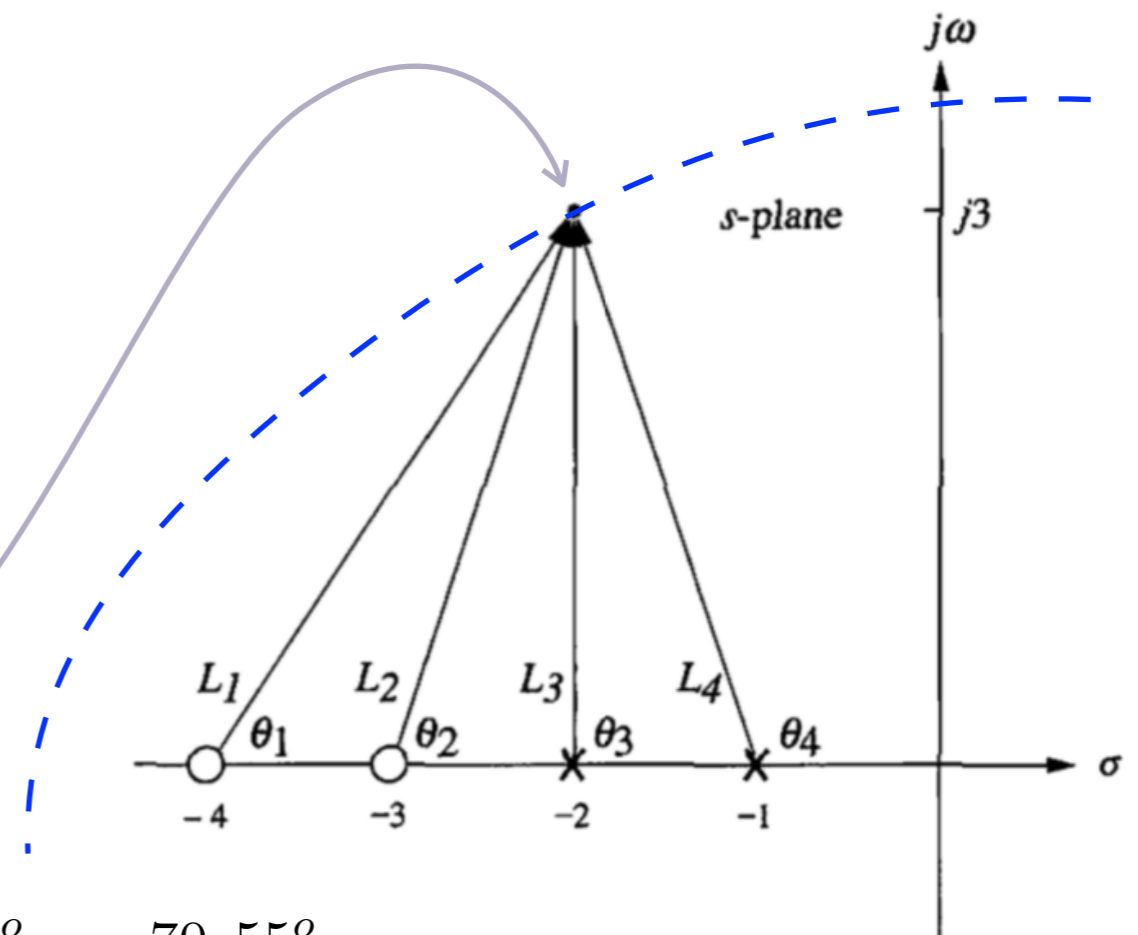
Somatório dos ângulos =  $180^\circ$ :

$$\sum \angle(\text{Zeros}) - \sum \angle(\text{Polos}) = \text{No. Ímpar} \cdot 180^\circ$$

Exemplo: ponto  $s = -2 + j3$  pertence ao RL  
(para certo valor de  $K$ ) !?

$$\theta_1 + \theta_2 - (\theta_3 + \theta_4) = 56,31^\circ + 71,57^\circ - 90^\circ - 108,43^\circ = -70,55^\circ$$

$\Rightarrow$  Conclusão: Não pertence ao RL (não pode ser um pólo de MF).



VER

$$FTMF(s) = \frac{K \cdot G(s)}{1 + K \cdot G(s)H(s)}$$

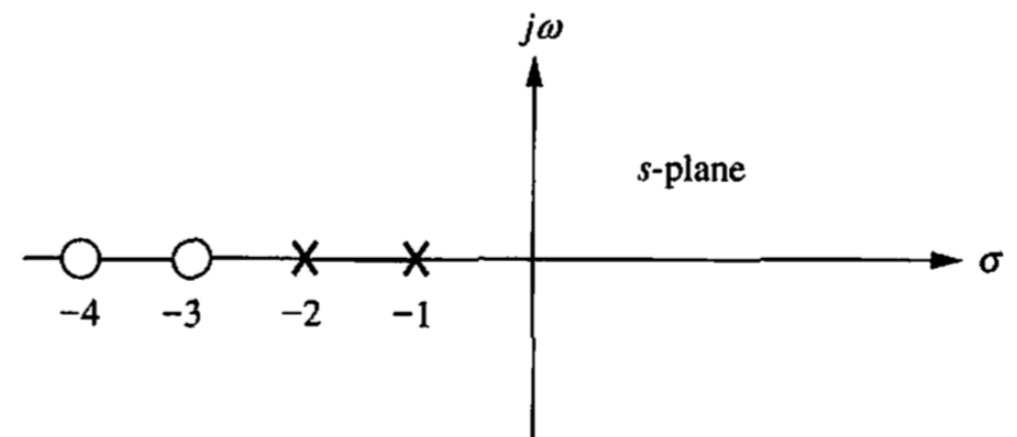
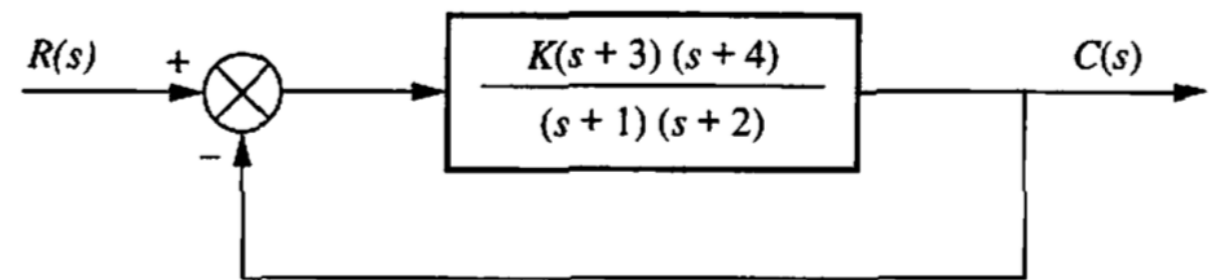
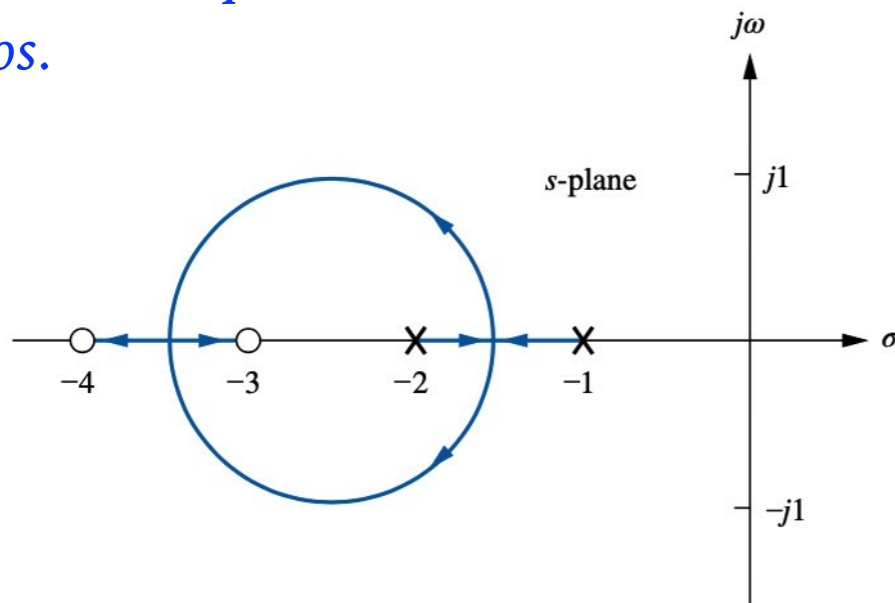
$$EC(z) = 1 + K \cdot G(s)H(s) = 0$$

$$K \cdot G(s)H(s) = -1 = 1 \angle [(2k + 1) \cdot 180^\circ], \quad \text{onde: } k = 0, \pm 1, \pm 2, \pm 3, \dots$$

$$|K \cdot G(s)H(s)| = 1$$

$$\angle KG(s)H(s) = (2k + 1) \cdot 180^\circ$$

No eixo real, para  $K > 0$ , o RL existe à esquerda de um número ímpar de pólos de MA finitos (no eixo real) e/ou à esquerda de um número de zeros de MA finitos.



De acordo com a regra, os segmentos do eixo real do RL estão entre -1 e -2 e entre -3 e -4.

# PROPRIEDADES (REGRAS) DO ROOT LOCUS (RL)

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$$FTMF(s) = \frac{K \cdot G(s)}{1 + K \cdot G(s)H(s)}$$

$$EC(z) = 1 + K \cdot G(s)H(s) = 0$$

$$K \cdot G(s)H(s) = -1 = 1 \angle [(2k + 1) \cdot 180^\circ], \quad \text{onde: } k = 0, \pm 1, \pm 2, \pm 3, \dots$$

$$|K \cdot G(s)H(s)| = 1$$

$$\angle KG(s)H(s) = (2k + 1) \cdot 180^\circ$$

*Ponto de partida da assíntota:*

$$\sigma_a = \frac{\sum \text{Polos finitos} - \sum \text{zeros finitos}}{\text{No. Polos finitos} - \text{No. Zeros finitos}}$$

*Ângulo de partida das assíntotas:*

$$\theta_a = \frac{(2k + 1) \cdot 180^\circ}{\text{No. Polos finitos} - \text{No. Zeros finitos}}, \quad \text{onde: } k = 0, \pm 1, \pm 2, \pm 3, \dots$$



# PROPRIEDADES (REGRAS) DO ROOT LOCUS (RL)

*Ponto de partida da assíntota:*

$$\sigma_a = \frac{\sum \text{Polos finitos} - \sum \text{zeros finitos}}{\text{No. Polos finitos} - \text{No. Zeros finitos}}$$

*Ângulo de partida das assíntotas:*

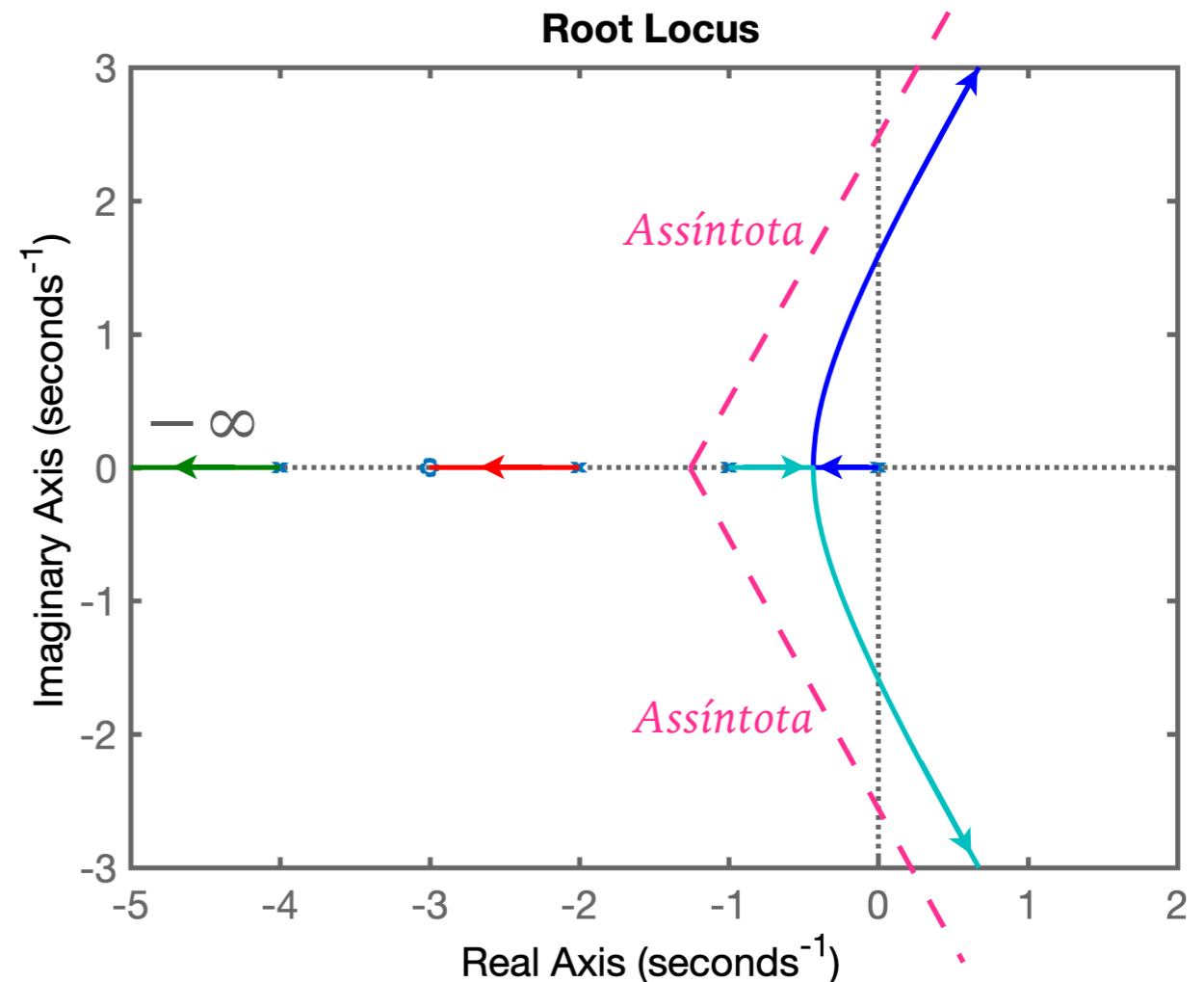
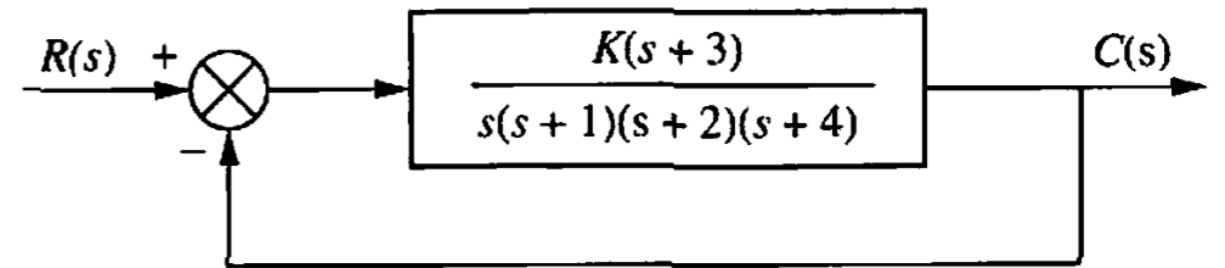
$$\theta_a = \frac{(2k + 1) \cdot 180^\circ}{\text{No. Polos finitos} - \text{No. Zeros finitos}}, \quad \text{onde: } k = 0, \pm 1, \pm 2, \pm 3, \dots$$

$$\sigma_a = \frac{(-1 - 2 - 4) - (-3)}{4 - 1} = -\frac{4}{3}$$

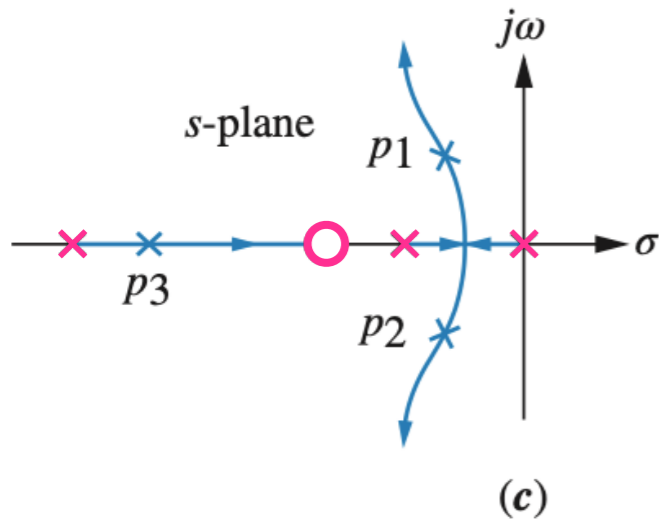
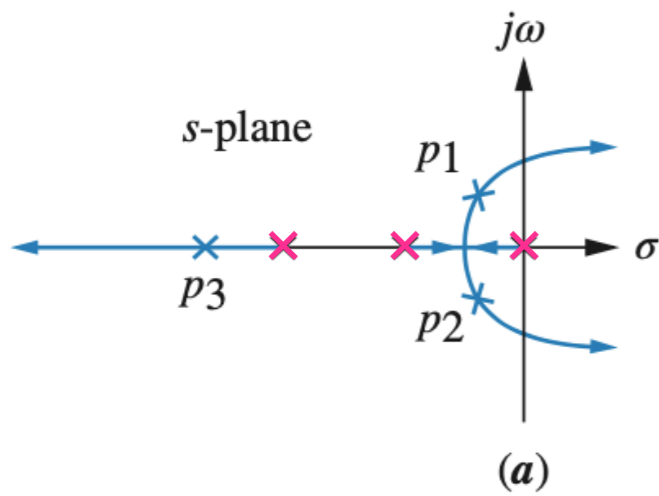
$$\theta_a = \frac{(2k + 1) \cdot 180^\circ}{4 - 1} = \frac{1 \cdot 180^\circ}{3} = 60^\circ, \quad \text{para } k = 0;$$

$$= \frac{3 \cdot 180^\circ}{3} = 0^\circ, \quad \text{para } k = 1;$$

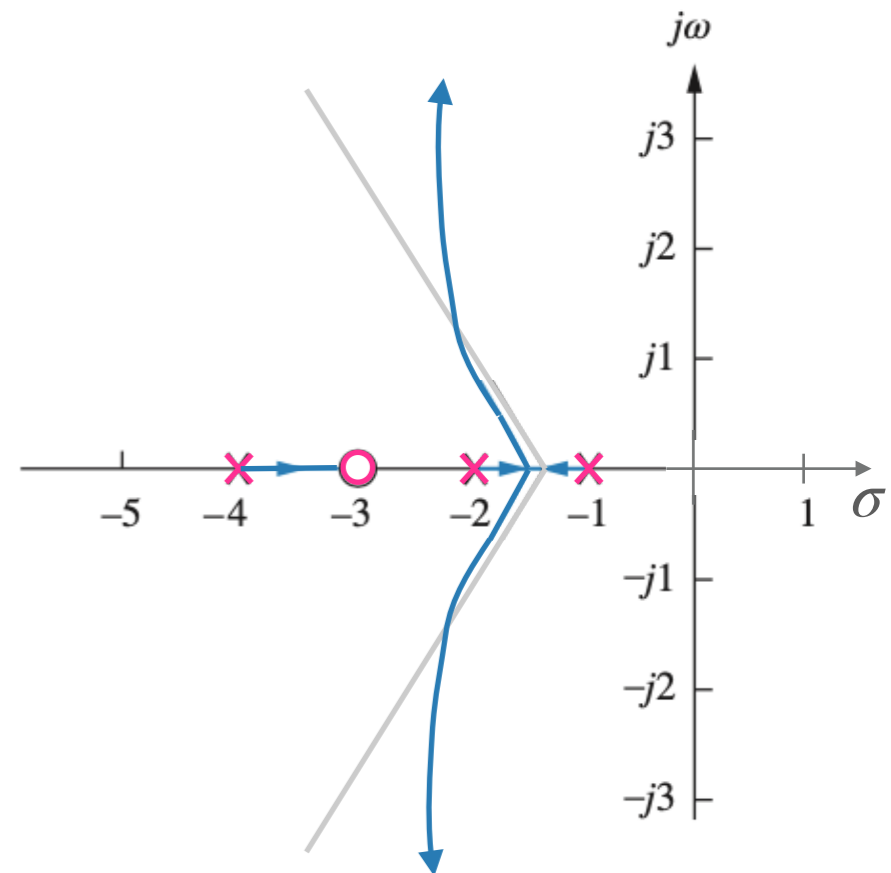
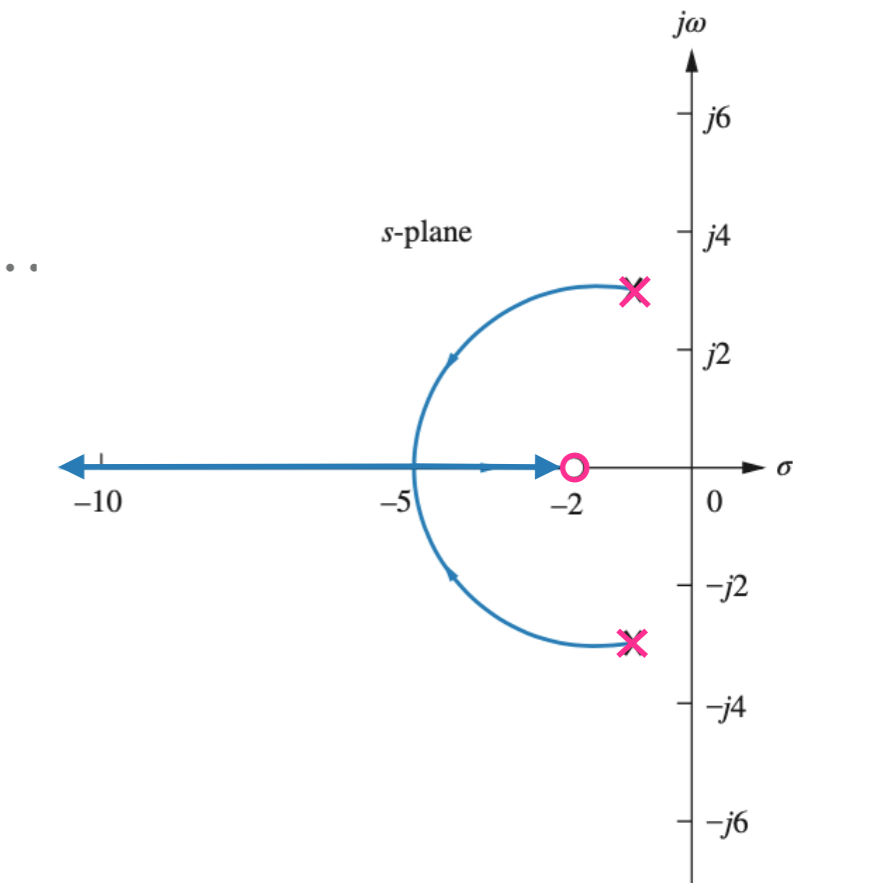
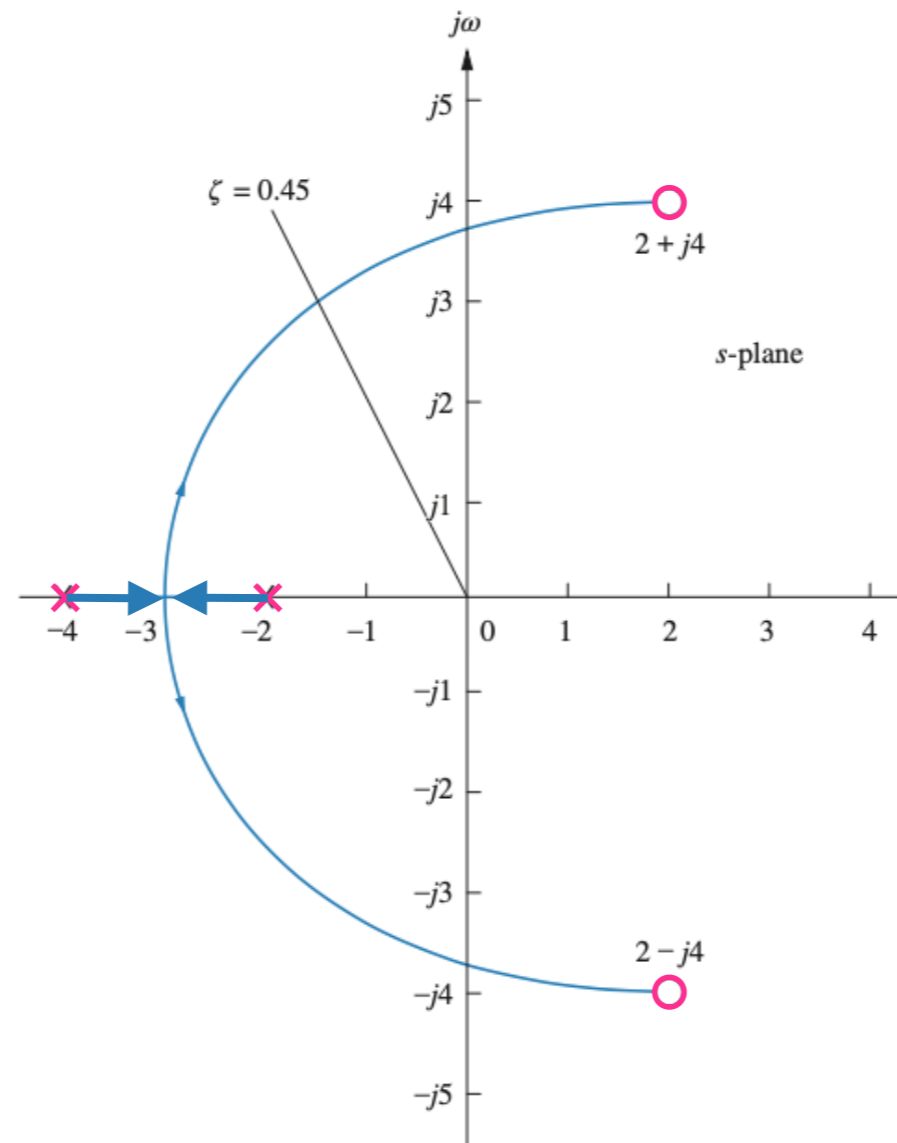
$$= \frac{5 \cdot 180^\circ}{5} = 300^\circ = -60^\circ, \quad \text{para } k = 2;$$



# ALGUNS EXEMPLOS DE RL



- x pólo de MA
- x pólo de MF
- o zero de MA



# REVISANDO...

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- RL × respostas temporais de sistemas com múltiplos pólos simples reais;
- RL × respostas temporais de sistemas de 2<sup>a</sup>-ordem (pólos complexos conjugados);
- Equações típicas prevendo respostas de sistemas de 2<sup>a</sup>-ordem com pólos complexos (sub-amortecidos);
- Linhas guias no RL para valores constantes de:
  - $\sigma$  (parte real)
  - $\omega$  (parte imaginária)
  - %OS (overshoot)
  - $\zeta$  (fator amortecimento)
  - $t_s$  (tempo de assentamento)
  - $t_p$  (tempo do pico).

*Ref.Bib.: Control Systems Engineering, Norman S. Nise (7a-ed), Cap. 8: Root Locus Techniques, 2015.*