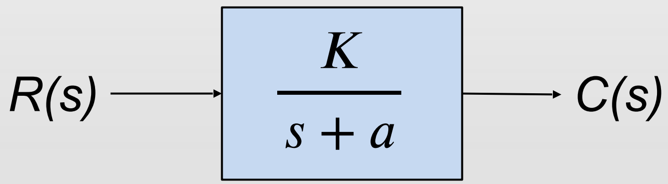
Respostas de Sistemas Lineares
Temporal x plano-s

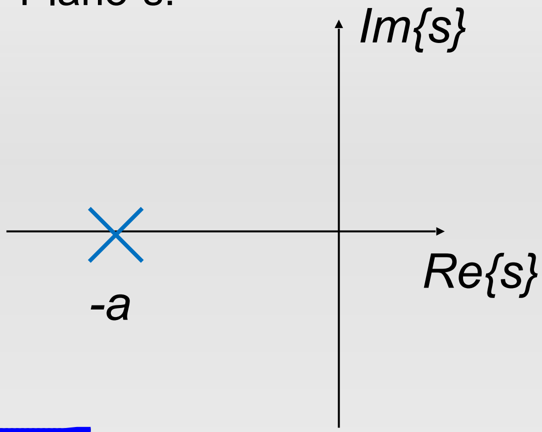
Controle Automático I
Prof. Fernando Passold

Versão Mai 2022

Respostas transitórias
Sistema de 1ª-ordem



Plano-s:



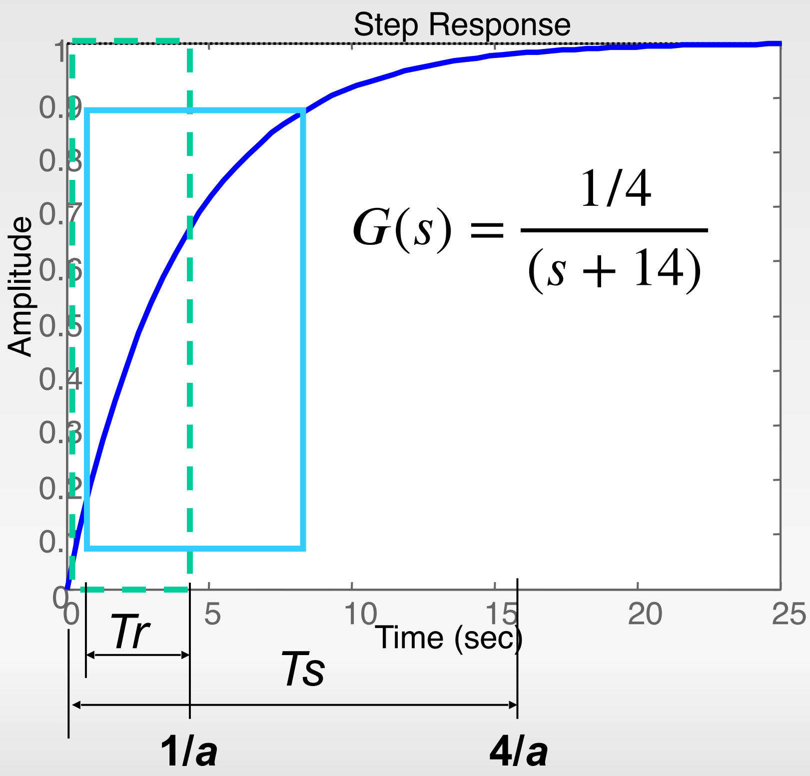
Resposta em malha fechada:

$$C(s) = R(s)G(s)$$

$$C(s) = \frac{1}{s} \cdot \frac{K}{(s+a)}$$

$$C(s) = \frac{K/a}{s} - \frac{K/a}{(s+a)}$$

$$c(t) = \frac{K}{a} - \frac{K}{a}e^{-at}$$

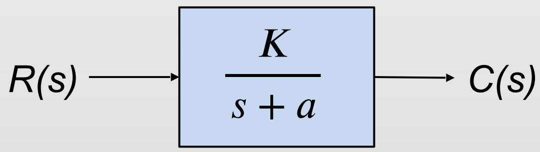


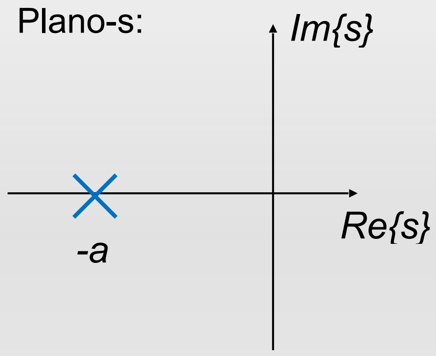
Constante de tempo → $\tau = \frac{1}{a}$

$$Tr = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$ ← Tempo de subida: K/a*[0.1 ~ 0.9]

$$Ts = \frac{4}{a}$$ ← Tempo de assentamento: K/a*0.98

Respostas transitórias
Sistema de 1ª-ordem





Resposta em malha fechada:

$$c(t) = \frac{K}{a} - \frac{K}{a}e^{-at}$$

Constante de tempo → $\tau = \frac{1}{a}$

Tempo de assentamento → $t_s = \frac{4}{a} = 4\,\tau$

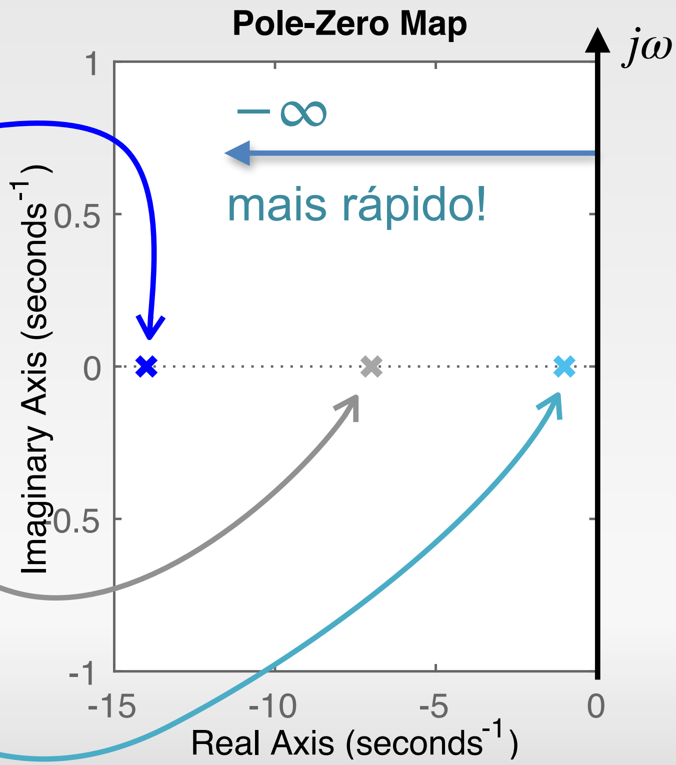
(= K/a*0,98)

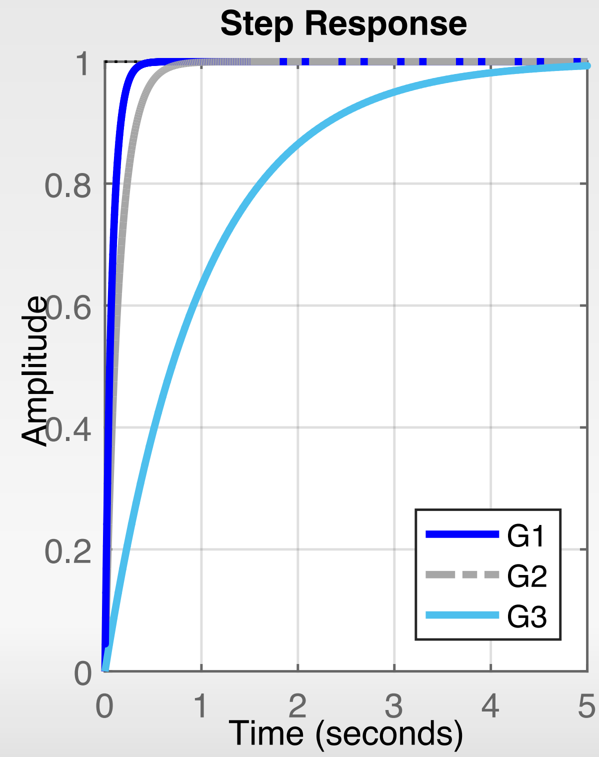
Comparando 3 sistemas de pólos simples reais:

$$G_1(s) = \frac{14}{s+14}$$

$$G_2(s) = \frac{7}{s+7}$$

$$G_3(s) = \frac{1}{s+1}$$





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Sistema de 1ª-ordem

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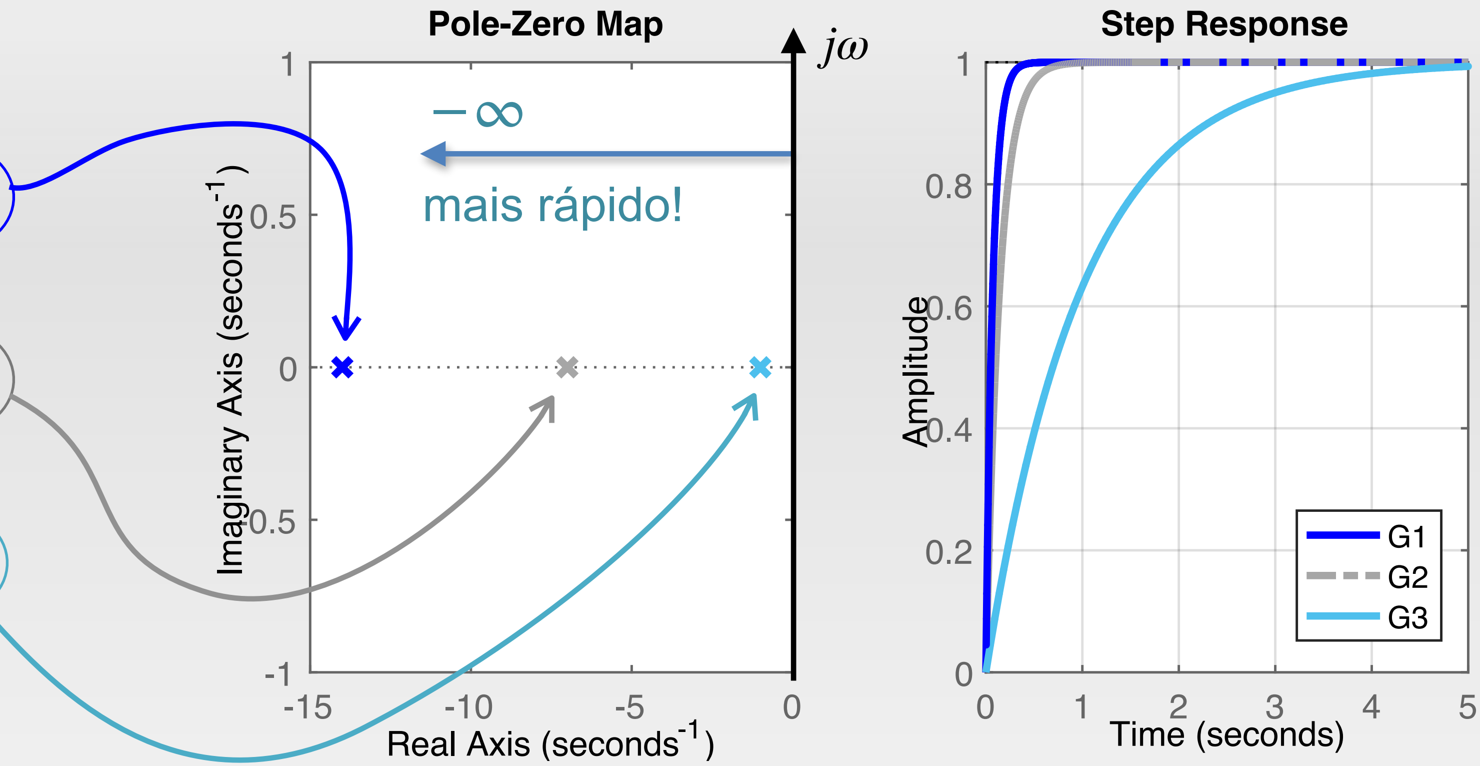
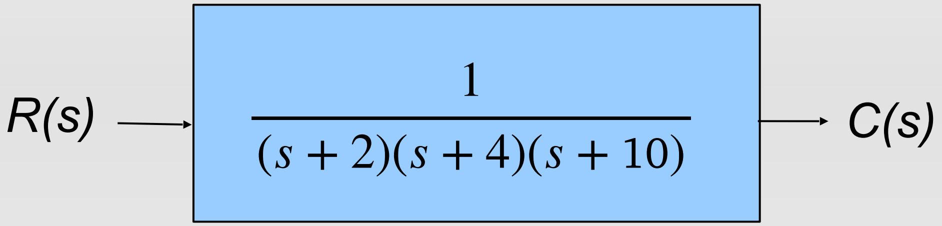



```
Matlab:
>> G1=tf(14,[1 14])

G1 =

    14
  ------
  s + 14

Continuous-time transfer function.
```

```
>> G2=tf(7, [1 7]);
>> G3=tf(1, [1 1]);
>> figure; subplot(121); pzmap(G1,G2,G3);
>> subplot(122); step(G1,G2,G3);
>> axis([0 5 0 1])
```

Respostas transitórias
Sistema com pólos reais

$R(s) = \frac{1}{s} \leftarrow$ Degrau (unitário)



$$C(s) = \frac{1}{s(s+2)(s+4)(s+10)}$$

```
>> num=1;
>> den=poly([0 -2 -4
-10]);
>>
[r,p,k]=residue(num,den)
r =
   -0.0021
    0.0208
   -0.0313
    0.0125
p =
  -10.0000
   -4.0000
   -2.0000
         0
k =
     []
>>
```

Resposta em malha aberta:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

$$K_1 = \frac{1 \cdot s}{s(s+2)(s+4)(s+10)}\bigg|_{s \to 0} = \frac{1}{80} = 0{,}0125$$

$$K_2 = \frac{1 \cdot (s+2)}{s(s+2)(s+4)(s+10)}\bigg|_{s \to -2}$$

$$K_2 = \frac{1}{s(s+4)(s+10)}\bigg|_{s \to -2} = -\frac{1}{32} = -0{,}0312$$

$$K_3 = \frac{1}{s(s+2)(s+10)}\bigg|_{s \to -4} = \frac{1}{48} = 0{,}0208$$

$$K_4 = \frac{1}{s(s+2)(s+4)}\bigg|_{s \to -10} = -\frac{1}{480} = -0{,}0021$$

Respostas transitórias
Sistema com pólos reais

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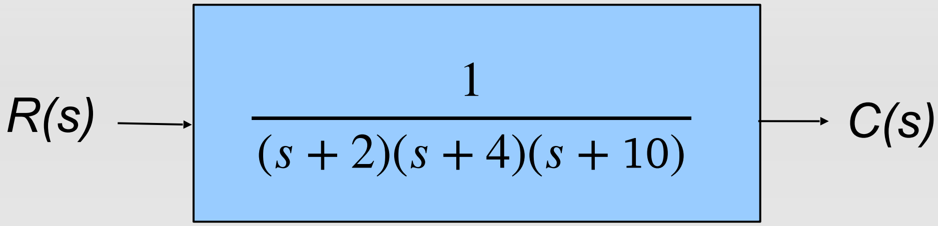


$$C(s) = \frac{1}{s(s+2)(s+4)(s+10)}$$

Resposta em malha aberta:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

$$K_1 = \frac{1 \cdot s}{s(s+2)(s+4)(s+10)}\Big|_{s\to 0} = \frac{1}{80} = 0{,}0125$$

```
>> num=1;
>> den=poly([0 -2 -4
-10]);
>>
[r,p,k]=residue(num,den)
r =
   -0.0021
    0.0208
   -0.0313
    0.0125
p =
  -10.0000
   -4.0000
   -2.0000
         0
k =
      []
>>
```

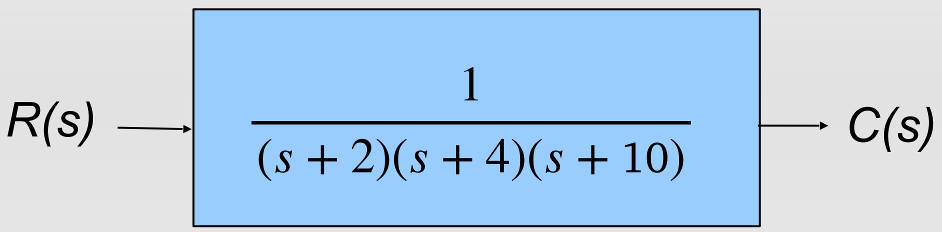
`[r,p,k] = residue(b,a)` finds the residues, poles, and direct term of a Partial Fraction Expansion of the ratio of two polynomials, where the expansion is of the form:

$$\frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{r_n}{s-p_n} + \ldots + \frac{r_2}{s-p_2} + \frac{r_1}{s-p_1} + k(s).$$

$$K_3 = \frac{}{s(s+2)(s+10)}\Big|_{s\to -4} = \frac{}{48} = 0{,}0208$$

$$K_4 = \frac{1}{s(s+2)(s+4)}\Big|_{s\to -10} = -\frac{1}{480} = -0{,}0021$$

Respostas transitórias
Sistema com pólos reais



Resposta em malha aberta:

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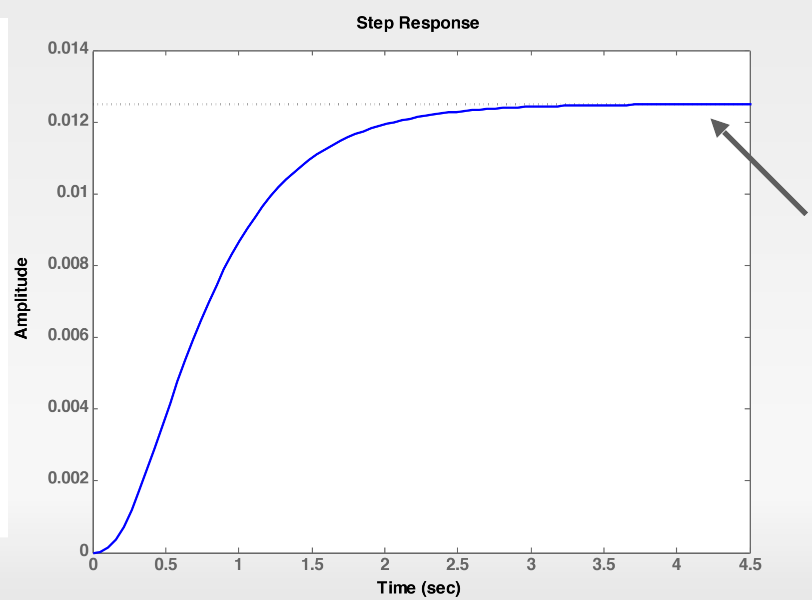
$$c(t) = K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-10t}$$

$$c(t) = \frac{1}{80} - \frac{1}{32} e^{-2t} + \frac{1}{48} e^{-4t} - \frac{1}{480} e^{-10t}$$

Matlab:

```
>> num=1;
>> den=poly([-2 -4 -10]);
>> c=tf(num,den);
>> zpk(c)
Zero/pole/gain:
         1
------------------
(s+10) (s+4) (s+2)

>> dcgain(c)
ans =     0.0125
>>
```



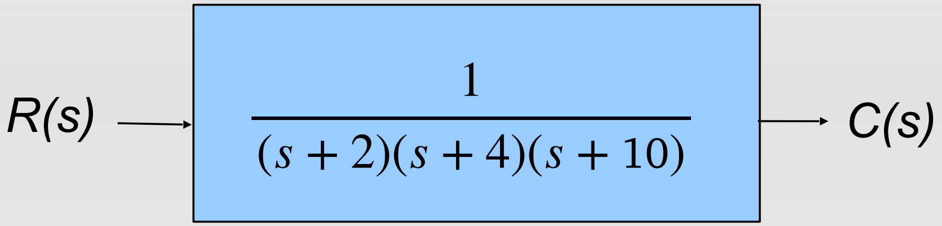
Valor final
(Teorema valor final $\mathscr{L}$):

$$c(\infty) = \lim_{s \to 0} s \cdot C(s)$$

$$c(\infty) = \frac{1 \cdot s}{s(s+2)(s+4)(s+10)}$$

$$c(\infty) = \frac{1}{80} = 0{,}0125$$

Respostas transitórias
Sistema com pólos reais



Resposta em malha aberta:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+10)}$$

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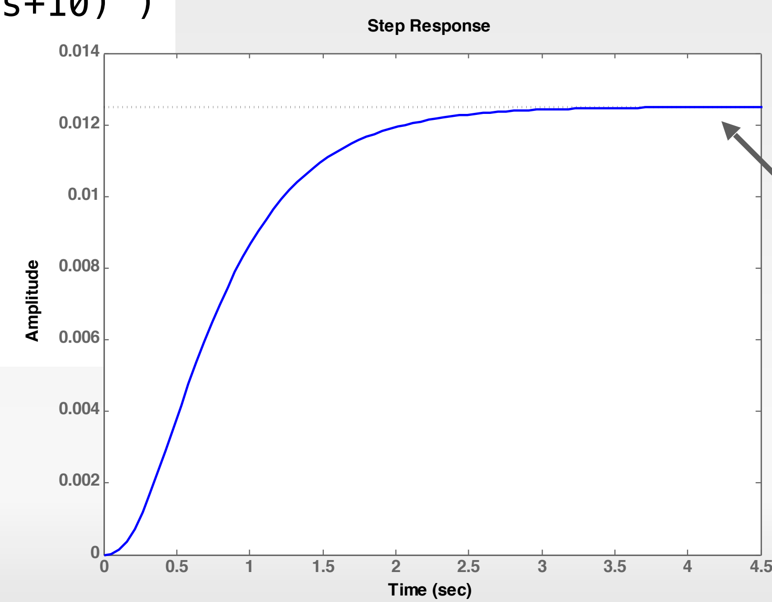
$$c(t) = \frac{1}{80} - \frac{1}{32} e^{-2t} + \frac{1}{48} e^{-4t} - \frac{1}{480} e^{-10t}$$

Matlab: outra forma para c(∞):

```
>> den=sym('(s+2)*(s+4)*(s+10)')
den =
(s+2)*(s+4)*(s+10)

>> subs(den,0)
ans =
    80

>>
```



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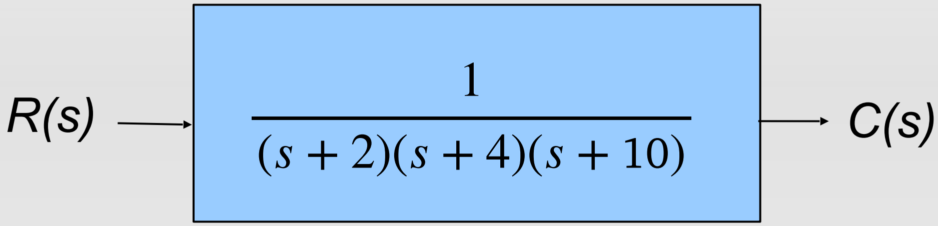

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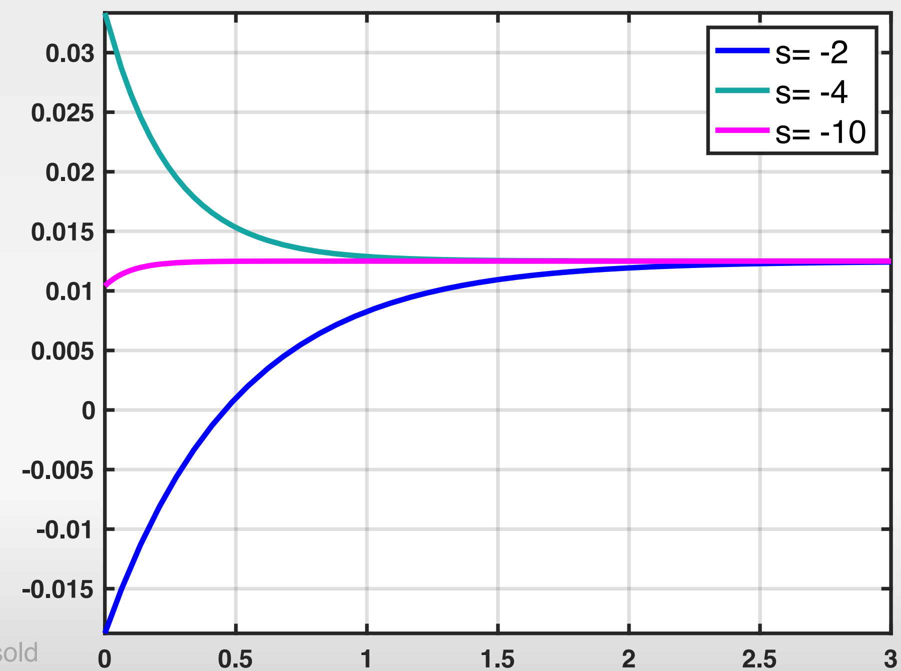
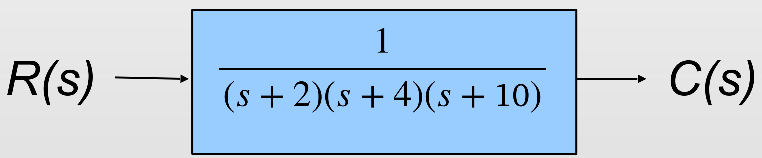
$$c(t) = \frac{1}{80} - \frac{1}{32} e^{-2t} + \frac{1}{48} e^{-4t} - \frac{1}{480} e^{-10t}$$

Matlab: verificando impacto de cada polo:

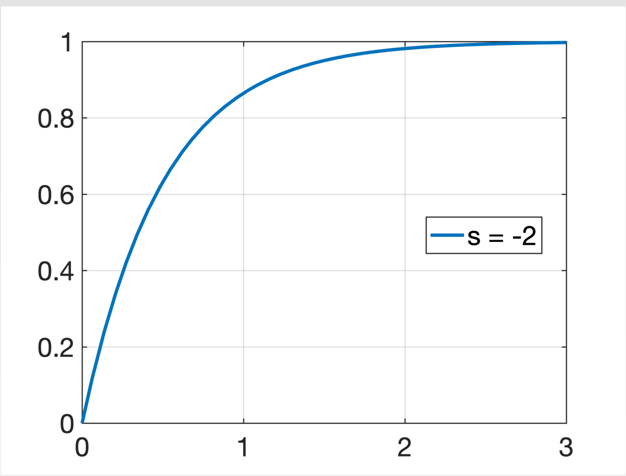
```
>> fplot(@(t)[(1/80)-(1/32)*exp(-2*t)],[0 3])
>> hold on
>> fplot(@(t)[(1/80)+(1/48)*exp(-4*t)],[0 3])
>> hold on
>> fplot(@(t)[(1/80)-(1/480)*exp(-10*t)],[0 3])
>> legend('s= -2','s= -4','s= -10')
```

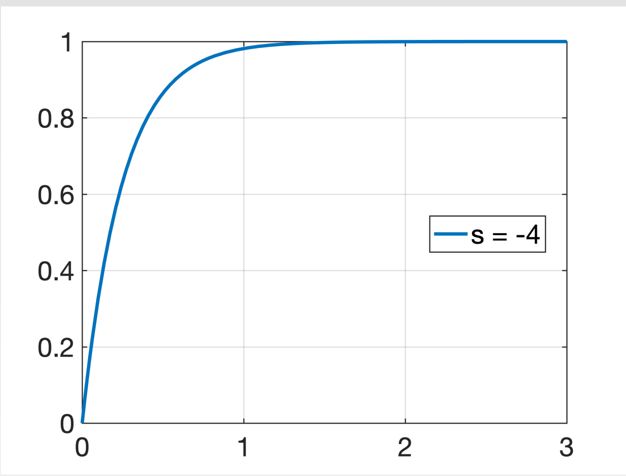


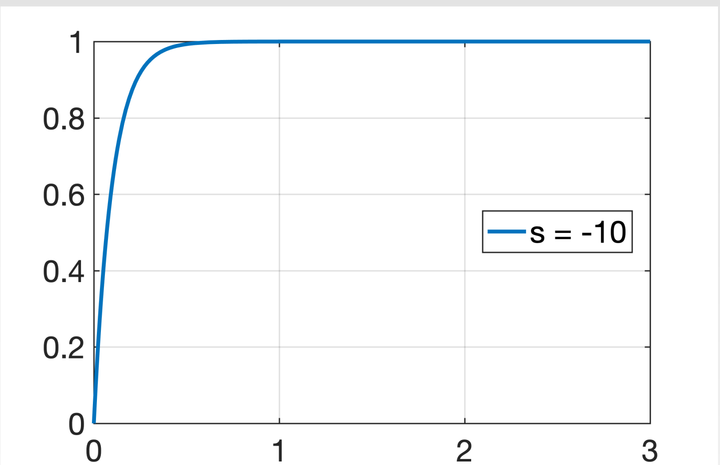
Respostas transitórias
Sistema com pólos reais



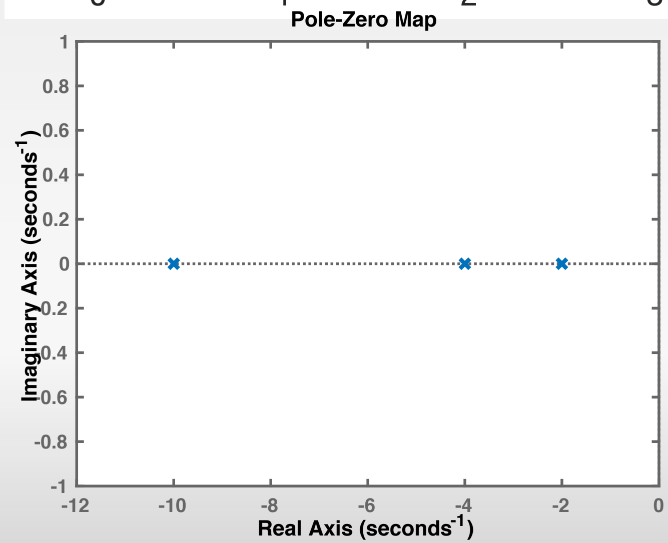
Avaliando as respostas de cada pólo individualmente:








```
>> figure; fplot(@(t)[(1-exp(-2*t))],[0 3])
>> legend('s = -2')
>> figure; fplot(@(t)[(1-exp(-4*t))],[0 3])
>> legend('s = -4')
>> figure; fplot(@(t)[(1-exp(-10*t))],[0 3])
>> legend('s = -10')
>> figure; pzmap(c)
```



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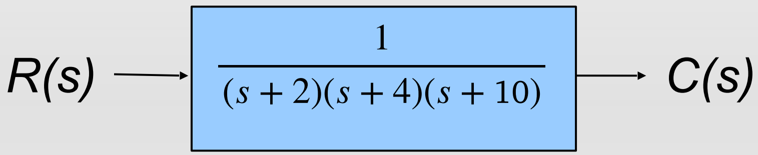

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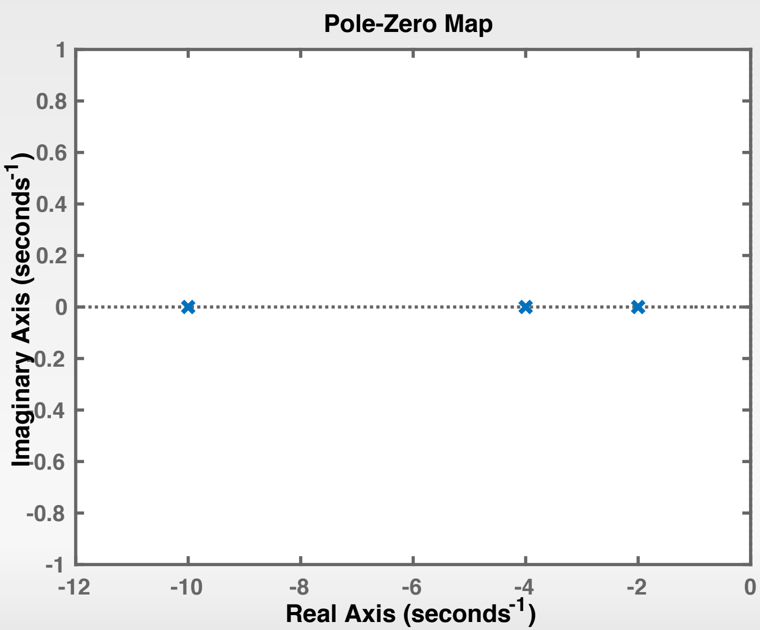
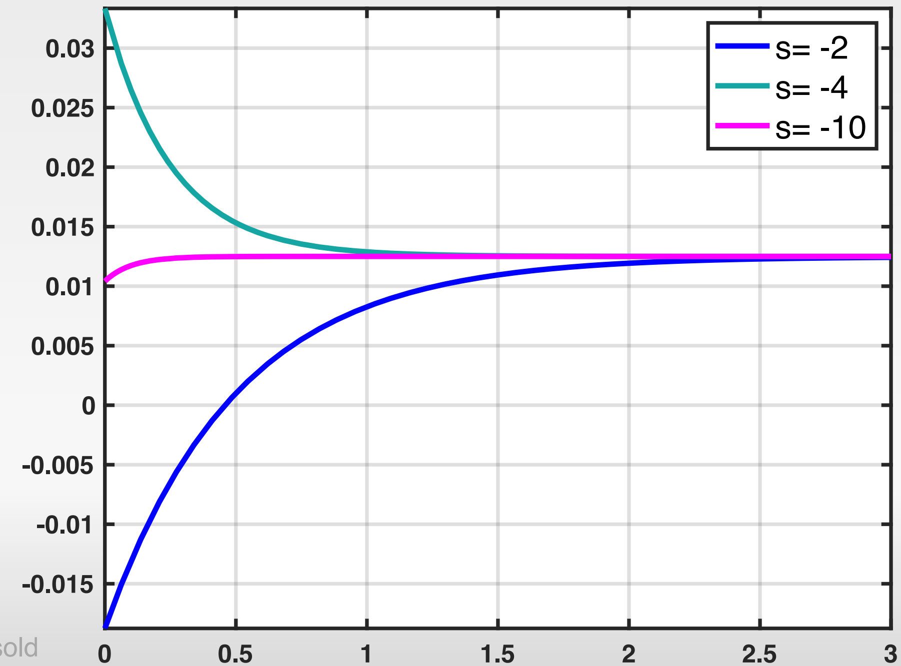
Matlab:

```
>> figure(3); pzmap(c)
```





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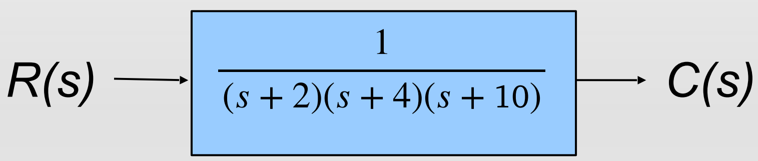

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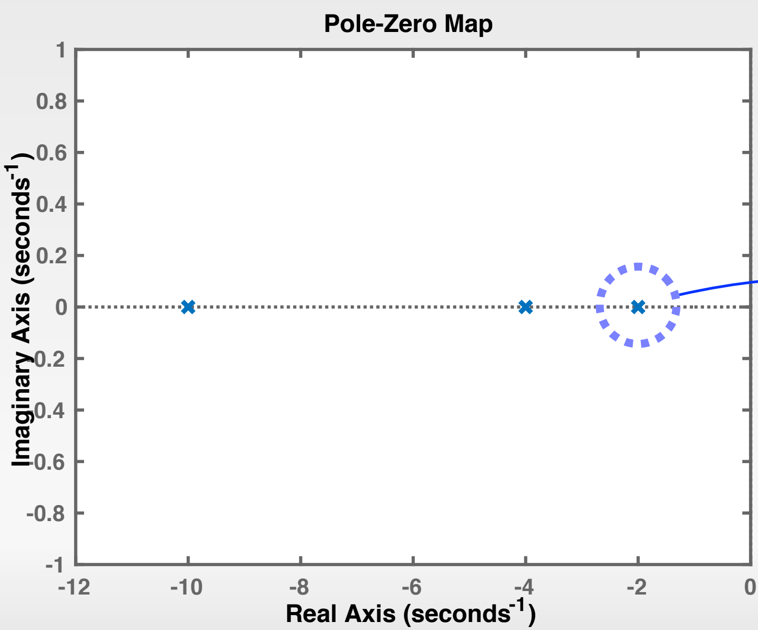
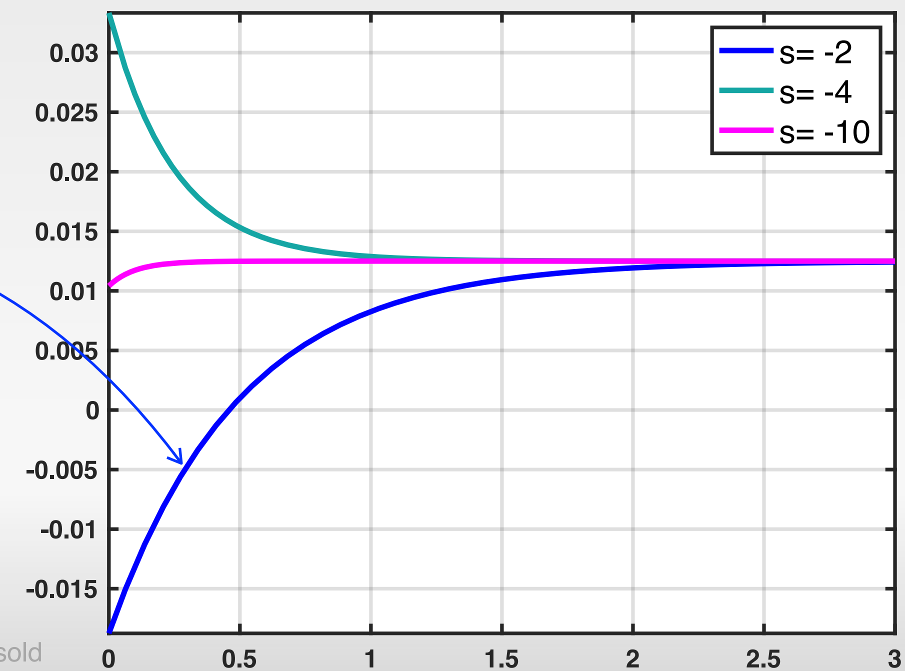
```





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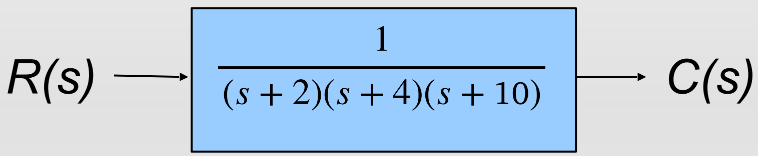

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```

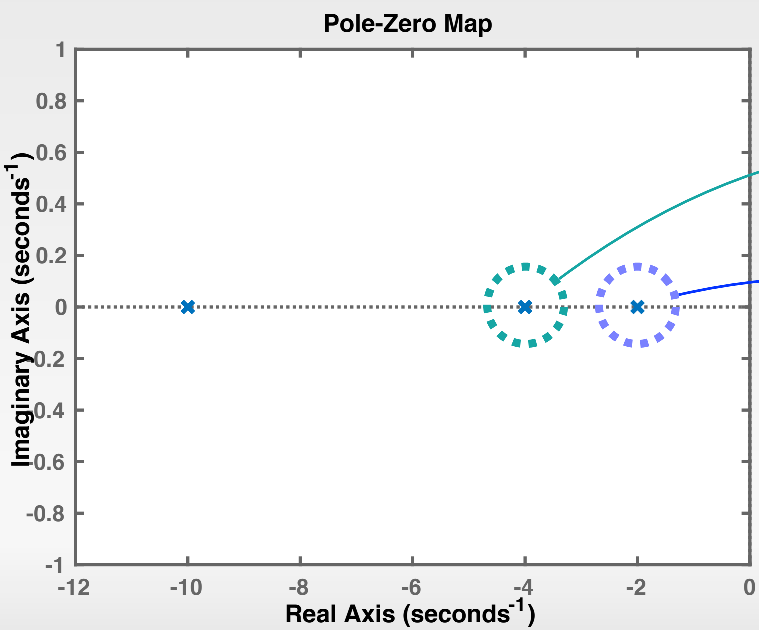

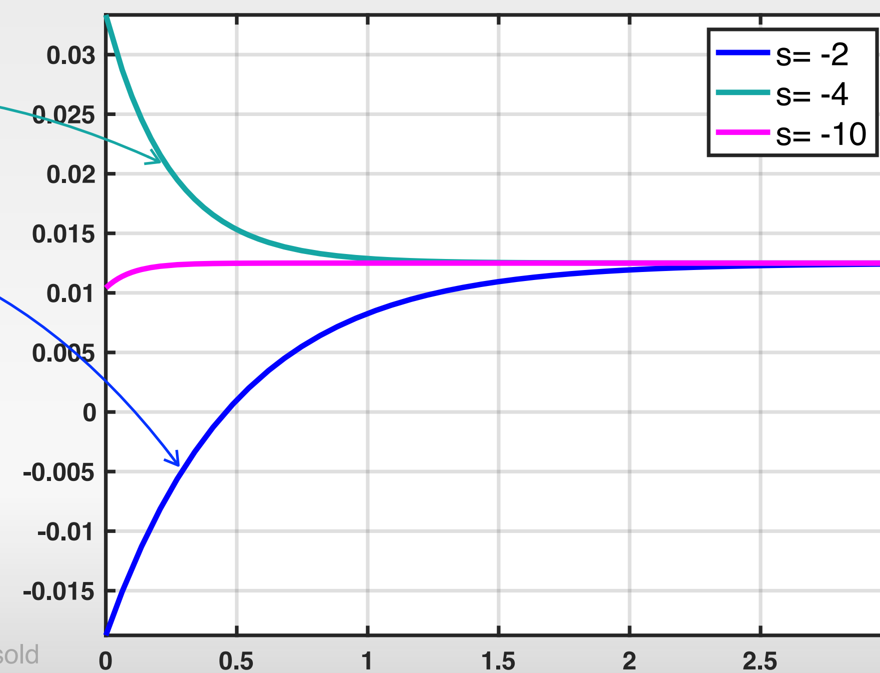

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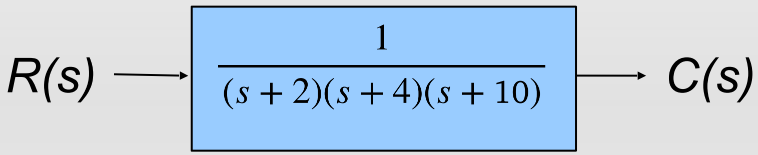
Sistema com pólos reais



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Matlab:

```
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```
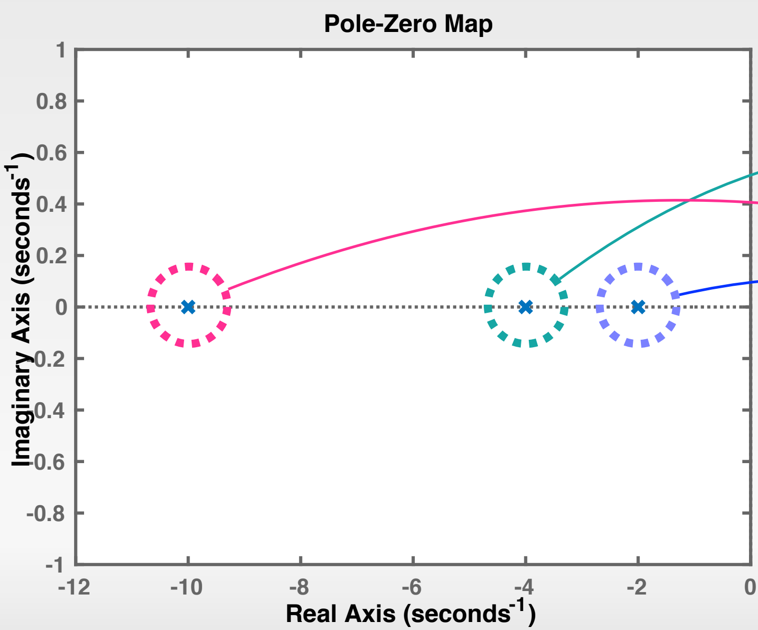
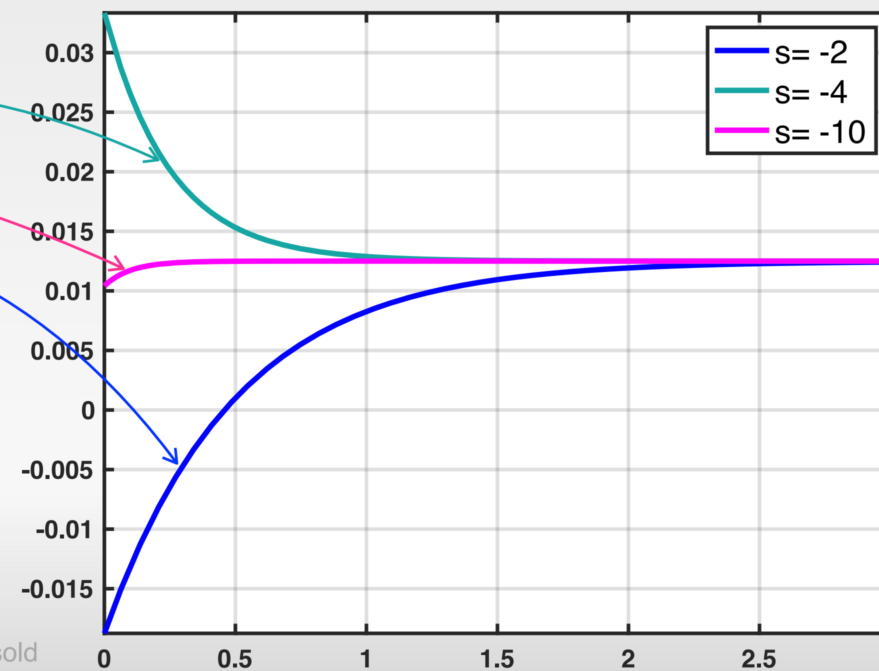
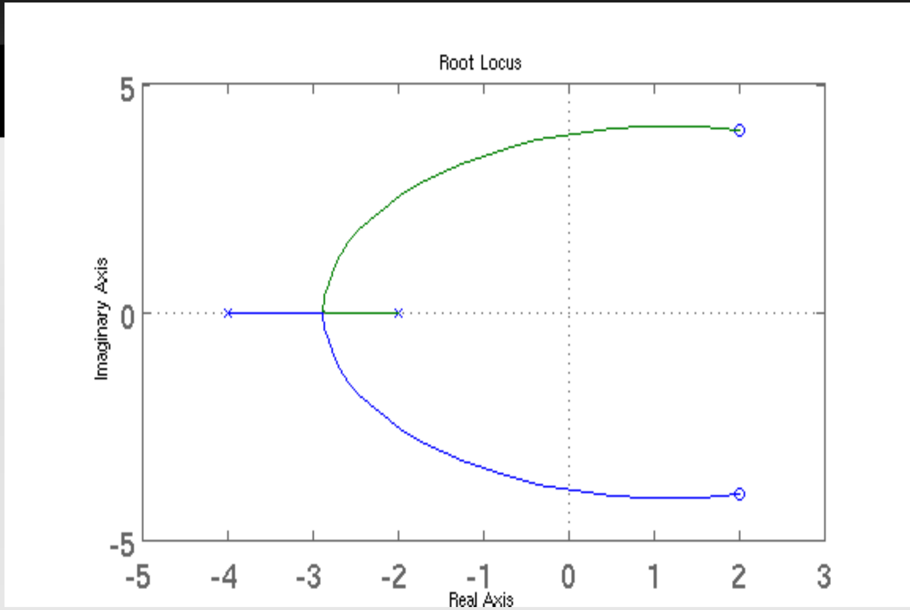





Respostas de Sistemas de 2ª ordem

Desenho do RL e encontrando pontos chaves



Exemplo: $$G(s) = \frac{K\left(s^2 - 4s + 20\right)}{(s+2)(s+4)}$$

$\zeta = 0.45$ $\quad$ $\theta = 180^o - arccos\left(0.45\right) = 116.7^o$

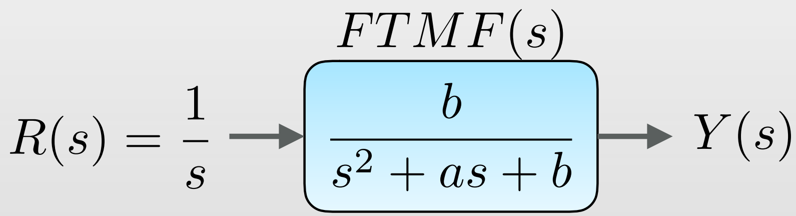
```
>> num=[1 -4 20];
>> den=conv([1 2],[1 4]);
>> g=tf(num,den)
Transfer function:
s^2 - 4 s + 20
--------------
s^2 + 6 s + 8

>> zpk(g)
Zero/pole/gain:
(s^2 - 4s + 20)
---------------
  (s+4) (s+2)

>> roots(num)
ans =
   2.0000 + 4.0000i
   2.0000 - 4.0000i

>> rlocus(g)
>>
```

Questões:

- Cruzamento com eixo $j\omega$
- Ponto de quebra
- Faixa de K (ganho):

Respostas transitórias

Sistemas de 2a-ordem:



$$FTMF(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{b}{s^2 + as + b}$$

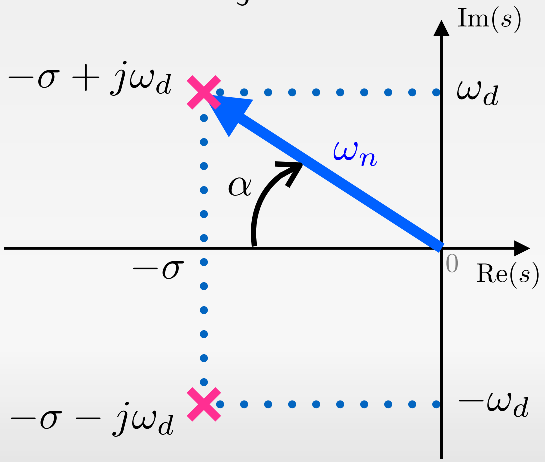
$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = K\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$

pólos em: $s = \sigma \pm j\omega_d$ ou: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

3 casos:

1) Raízes reais:
   a) Distintas;
   b) Iguais;
2) Raízes complexas:
   a) Distante de $j\omega$;
   b) Sobre $j\omega$.

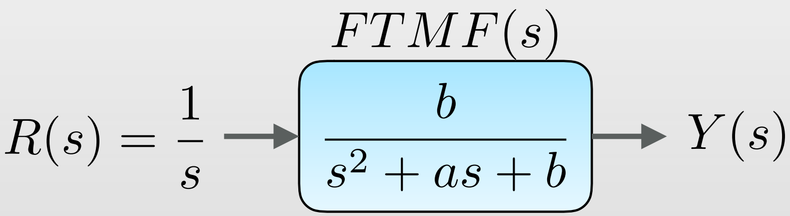
$$0 < \zeta < 1$$



$$\begin{aligned} \sigma &= \omega_n \cos(\alpha) = \omega_n \zeta; \\ \omega_d &= \omega_n \sin(\alpha) = \omega_n \sqrt{1-\zeta^2}; \\ \zeta &= \cos(\alpha); \\ \sin(\alpha) &= \sqrt{1-\zeta^2}; \end{aligned}$$

Respostas transitórias

Sistemas de 2a-ordem:



3 casos:

1) Raízes reais:
   a) Distintas;
   b) Iguais
2) Raízes complexas:
   a) Distante de $j\omega$;
   b) Sobre $j\omega$.

$$G_{1a}(s) = \frac{9}{s^2 + 9s + 9}$$

2 pólos reais em $-p_1$ e $-p_2$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k\frac{w_n^2}{s^2 + 2\zeta w_n + w_n^2}$$

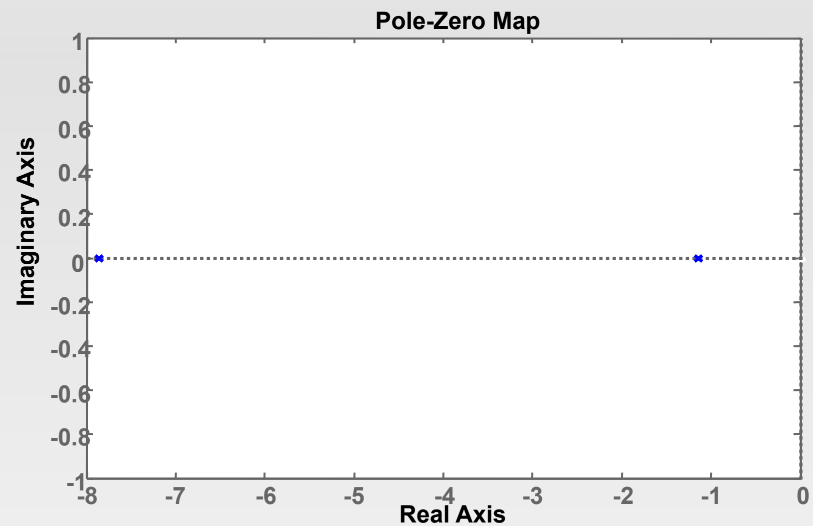
$$w_n^2 = 9 \rightarrow w_n = 3$$

$$\zeta = \frac{9}{6} = 1{,}5$$

$$y(t) = K_1 e^{-p_1 t} + K_2 e^{-p_2 t}$$

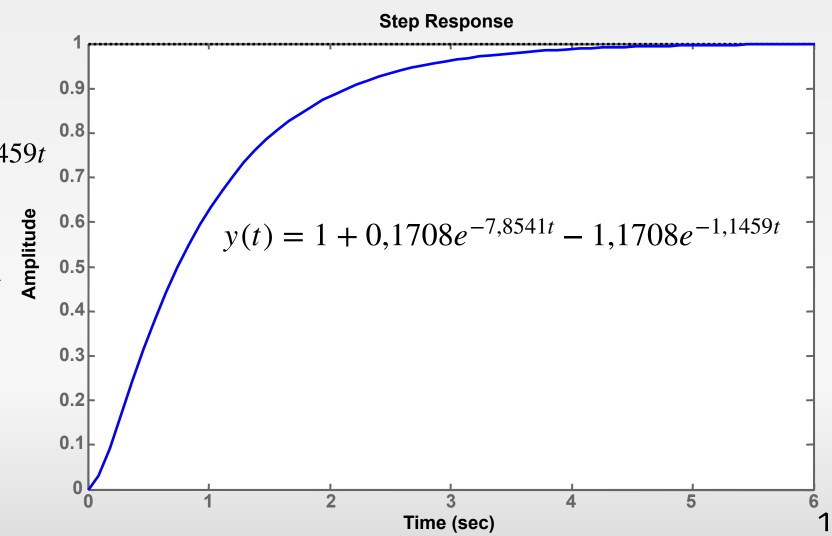
$$y(t) = 1 + 0{,}1708e^{-7{,}8541t} - 1{,}1708e^{-1{,}1459t}$$

Sistema **superamortecido** ➟ ($\zeta > 1$)

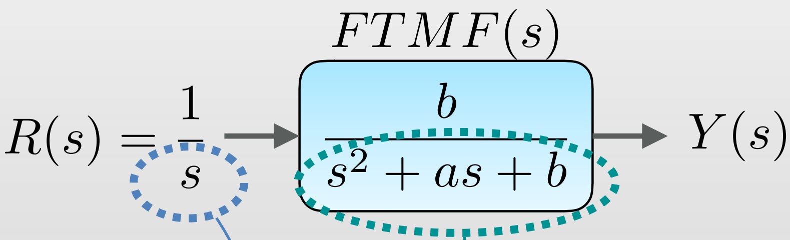


Sistema **superamortecido** ➟ ($\zeta > 1$)

Overdamped response ➟



Sistemas de 2a-ordem:



3 casos:

1) Raízes reais:
   a) Distintas;
   b) Iguais

Matlab:

```
>> num=9;
>> den=conv([1 0],[1 9 9]);
>> [r,p,k]=residue(num,den)
r =
    0.1708
   -1.1708
    1.0000
p =
   -7.8541
   -1.1459
         0
k =
     []
>> ftmf=tf( 9, [1 9 9]);
>> pzmap(ftmf)
>> step(ftmf)
```

$$G_{1a}(s) = \frac{9}{s^2 + 9s + 9}$$

2 pólos reais em $-p_1$ e $-p_2$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k\frac{w_n^2}{s^2 + 2\zeta w_n + w_n^2}$$

$$w_n^2 = 9 \rightarrow w_n = 3$$

$$\zeta = \frac{9}{6} = 1{,}5$$

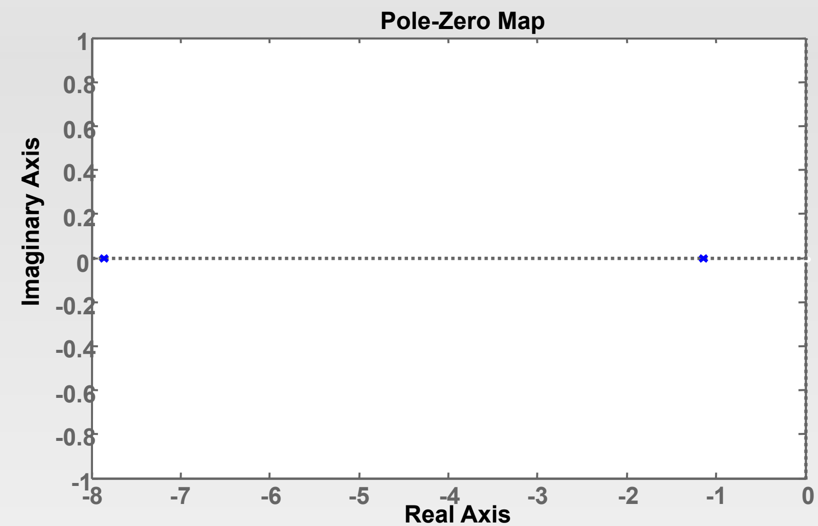
$$y(t) = K_1 e^{-p_1 t} + K_2 e^{-p_2 t}$$

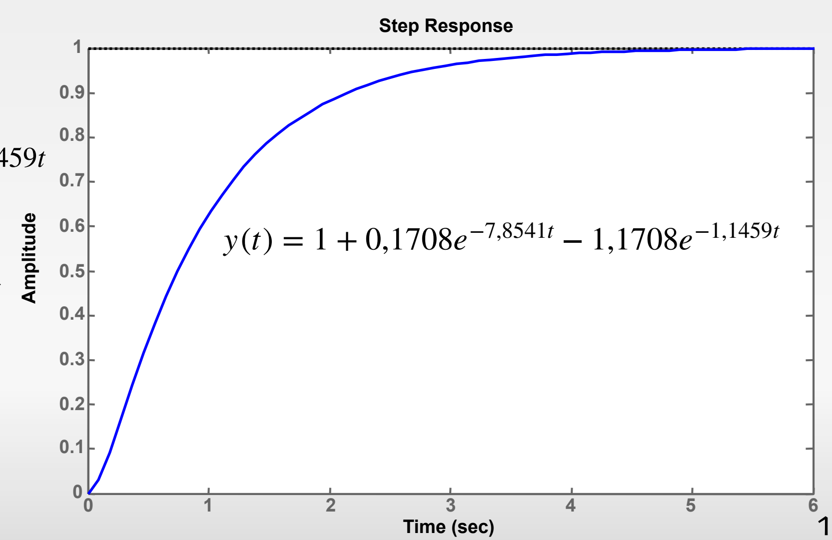
$$y(t) = 1 + 0{,}1708e^{-7{,}8541t} - 1{,}1708e^{-1{,}1459t}$$

Sistema **superamortecido** ➟ ($\zeta > 1$)

Overdamped response ➟

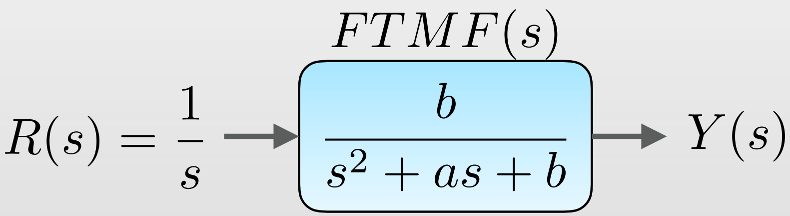
Sistema **superamortecido** ➟ ($\zeta > 1$)





Respostas transitórias

Sistemas de 2a-ordem:



3 casos:

1) Raízes reais:
    a) Distintas;
    b) Iguais
2) Raízes complexas:
    a) Distante de $j\omega$;
    b) Sobre $j\omega$.

$$G(s) = \frac{9}{s^2 + 6s + 9}$$

2 pólos reais em $-p_1$ e $-p_2$

$$p_{1,2} = -3$$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k\frac{w_n^2}{s^2 + 2\zeta w_n + w_n^2}$$

$$w_n^2 = 9 \rightarrow w_n = 3$$

$$\zeta = \frac{6}{6} = 1$$

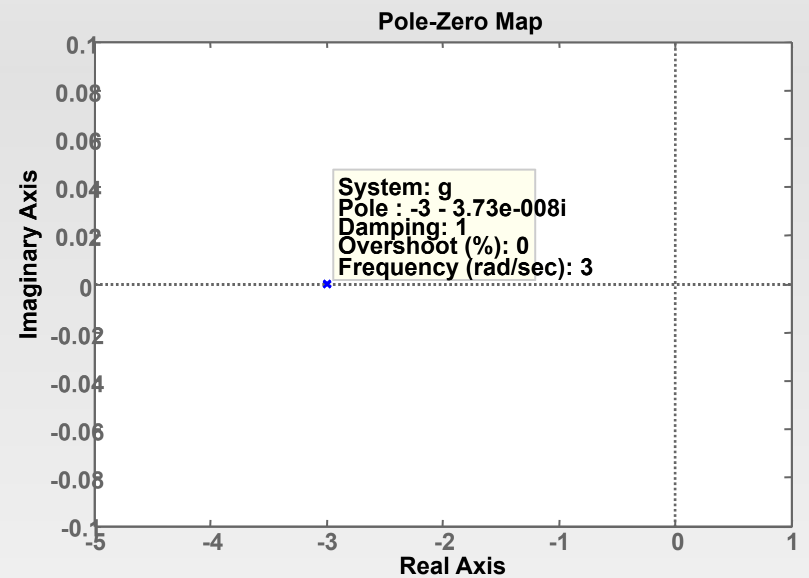
$$y(t) = K_0 + K_1 e^{-at} + K_2 t e^{-at}$$

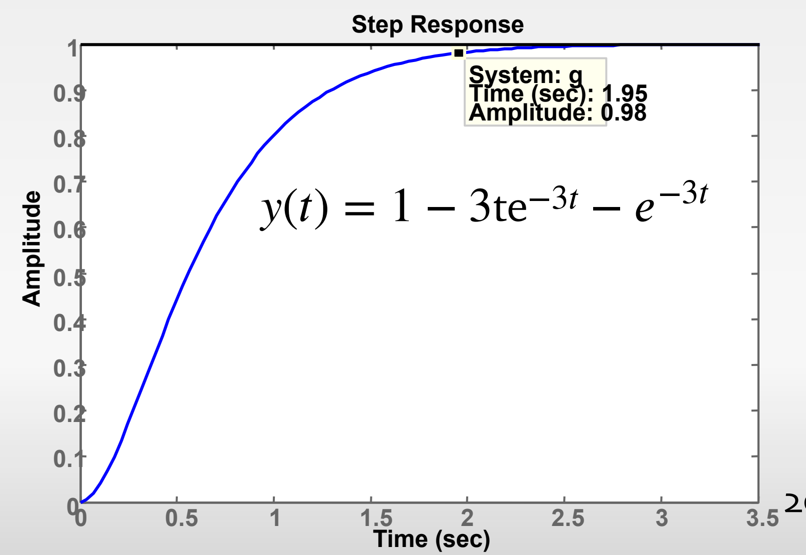
$$y(t) = 1 - 3te^{-3t} - e^{-3t}$$

Amortecimento crítico ➟ ($\zeta = 1$)

Critically damped ➟

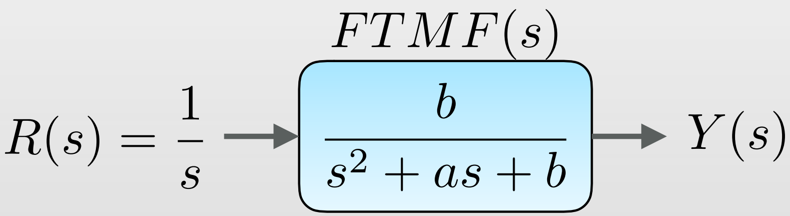
Amortecimento crítico ➟ ($\zeta = 1$)





Respostas transitórias

Sistemas de 2a-ordem:



3 casos:

1) Raízes reais:
   a) Distintas;
   b) Iguais

Matlab:

```
>> num=9;
>> den=conv([1 0],[1 6
9]);
>>
[r,p,k]=residue(num,den)
r =
    -1
    -3
     1
p =
    -3
    -3
     0
k =
     []
>> ftmf=tf( 9, [1 6 9]);
>> pzmap(ftmf)
>> step(ftmf)
```

$$G(s) = \frac{9}{s^2 + 6s + 9}$$

2 pólos reais em $-p_1$ e $-p_2$

$$p_{1,2} = -3$$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k\frac{w_n^2}{s^2 + 2\zeta w_n + w_n^2}$$

$$w_n^2 = 9 \rightarrow w_n = 3$$

$$\zeta = \frac{6}{6} = 1$$

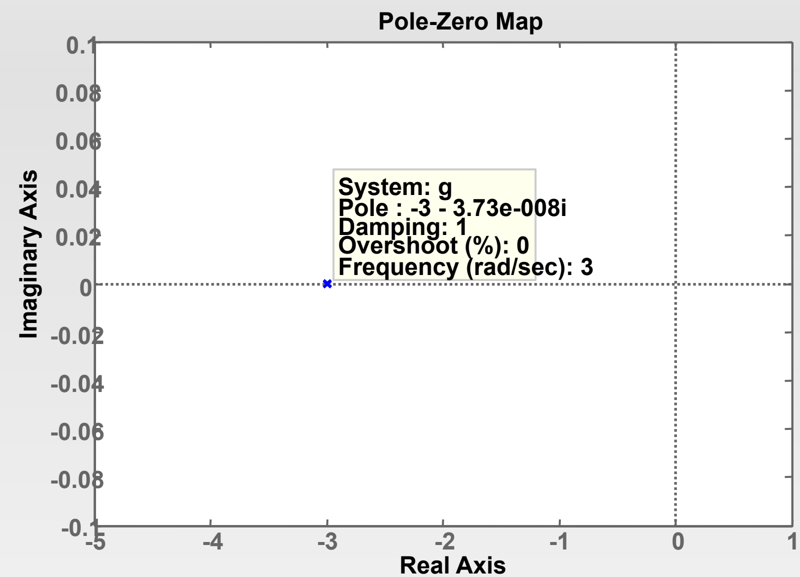
$$y(t) = K_0 + K_1 e^{-at} + K_2 t e^{-at}$$

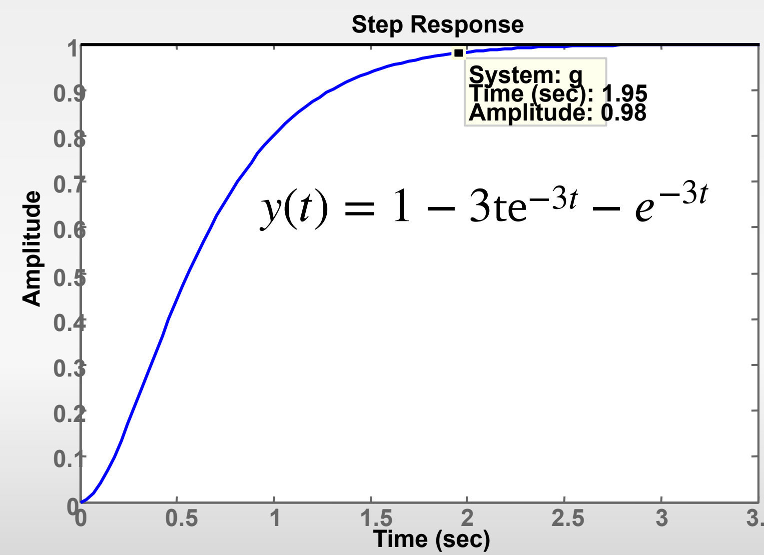
$$y(t) = 1 - 3\text{t}e^{-3t} - e^{-3t}$$

Amortecimento crítico ➟ ($\zeta = 1$)

Critically damped ➟

Amortecimento crítico ➟ ($\zeta = 1$)





Respostas transitórias

Sistemas de 2a-ordem:

$$R(s) = \frac{1}{s} \rightarrow \underset{FTMF(s)}{\boxed{\frac{b}{s^2 + as + b}}} \rightarrow Y(s)$$

Sistema **subamortecido** ➟ ($0 < \zeta < 1$)

3 casos:

1) Raízes reais:
   a) Distintas;
   b) Iguais
2) Raízes complexas:
   a) Distante de $j\omega$;
   b) Sobre $j\omega$.

$$G(s) = \frac{9}{s^2 + 2s + 9}$$

2 pólos complexos em $-p_1$ e $-p_2$

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k\frac{w_n^2}{s^2 + 2\zeta w_n + w_n^2}$$

$$p_{1,2} = \sigma \pm j w_d \qquad w_d = w_n \sin(\alpha) = w_n\sqrt{1-\zeta^2}$$

$$p_{1,2} = -1 \pm j\sqrt{8}$$

$$w_n^2 = 9 \rightarrow w_n = 3$$

$$\zeta = \frac{2}{6} = 0{,}3333$$

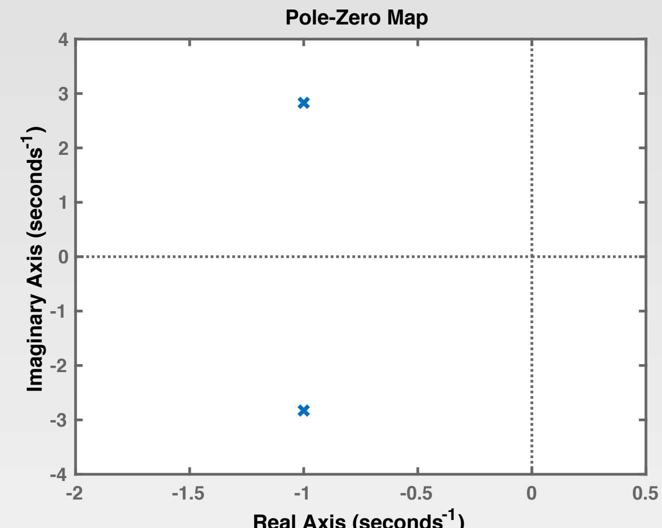
$$y(t) = 1 - e^{-t}[\cos(\sqrt{8}t) + \frac{\sqrt{8}}{8}\sin(\sqrt{8}t)]$$

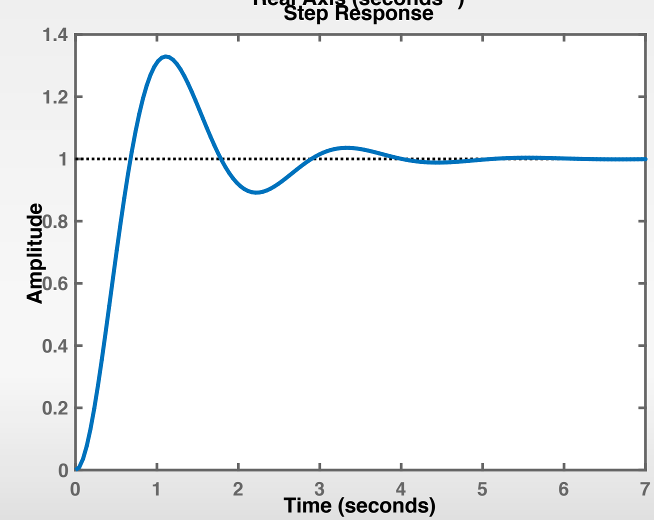
$$y(t) = 1 - 1{,}06e^{-t}\cos(\sqrt{8}t - 19{,}47^o)$$

$$y(t) = 1 - Ae^{-\sigma\% t}\cos(w_d t - \phi)$$

Sistema **subamortecido** ➟
($0 < \zeta < 1$)

underdamped response ➟





Respostas transitórias

Sistemas de 2a-ordem:

$FTMF(s)$

$$R(s) = \frac{1}{s} \rightarrow \boxed{\frac{b}{s^2 + as + b}} \rightarrow Y(s)$$

3 casos:
1) Raízes reais:
   a) Distintas;
   b) Iguais
2) Raízes complexas:
   a) Distante de $j\omega$;
   b) Sobre $j\omega$.

$$G(s) = \frac{9}{s^2 + 9}$$

2 pólos complexos em $\pm j\omega_n$:

$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = k\frac{w_n^2}{s^2+2\zeta w_n + w_n^2}$$

$$p_{1,2} = \pm j\sqrt{3}$$

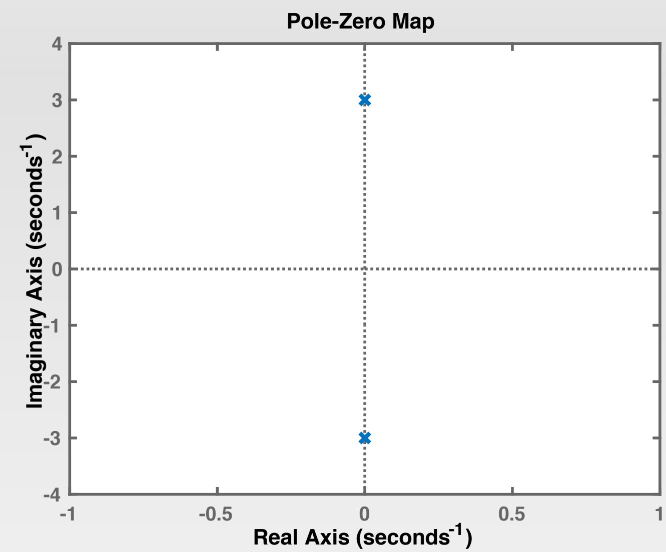
$$w_n^2 = 9 \rightarrow w_n = 3$$

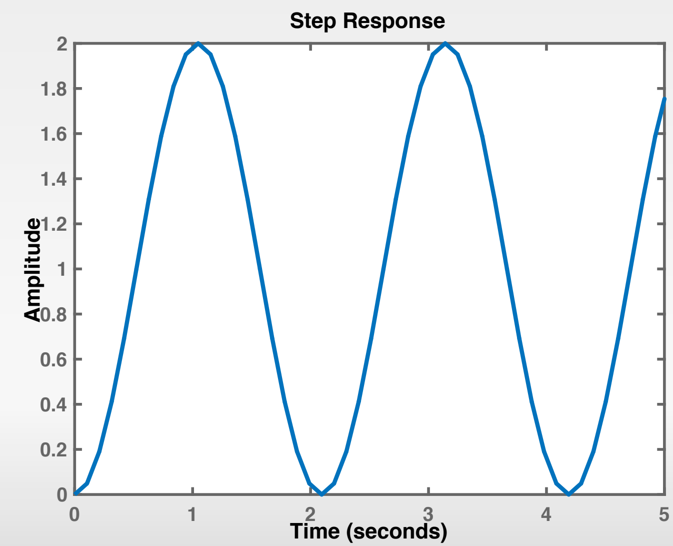
$$\zeta = 0$$

$$y(t) = 1 - \cos(3t)$$

$$y(t) = A\cos(w_1 t - \phi)$$

Sistema **oscilatório** ➟ ($\zeta = 0$)





Sistema **oscilatório** ➟ ($\zeta = 0$)

undamped response ➟

Respostas transitórias

Sistemas de 2a-ordem:

$$R(s) = \frac{1}{s} \rightarrow \underbrace{\frac{b}{s^2 + as + b}}_{FTMF(s)} \rightarrow Y(s)$$

- Super amortecido: $\zeta > 1$

$$\frac{9}{s^2 + 9s + 9}$$

```
>> pole(g1)
  -7.8541
  -1.1459
```

- **Subamortecido:** $0 < \zeta < 1$

$$\frac{9}{s^2 + 2s + 9}$$

```
>> pole(g2)
 -1.0000 + 2.8284i
 -1.0000 - 2.8284i
```

$$y(t) = 1 - \exp(-\zeta\omega_n t) \cdot \left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t)\right]$$

$$\text{pólos em: } s = \sigma \pm j\omega_d \quad \begin{cases} \sigma = \omega_n \zeta; \\ \omega_d = \omega_n \sqrt{1-\zeta^2}; \end{cases}$$

- Oscilatório: $\zeta = 0$

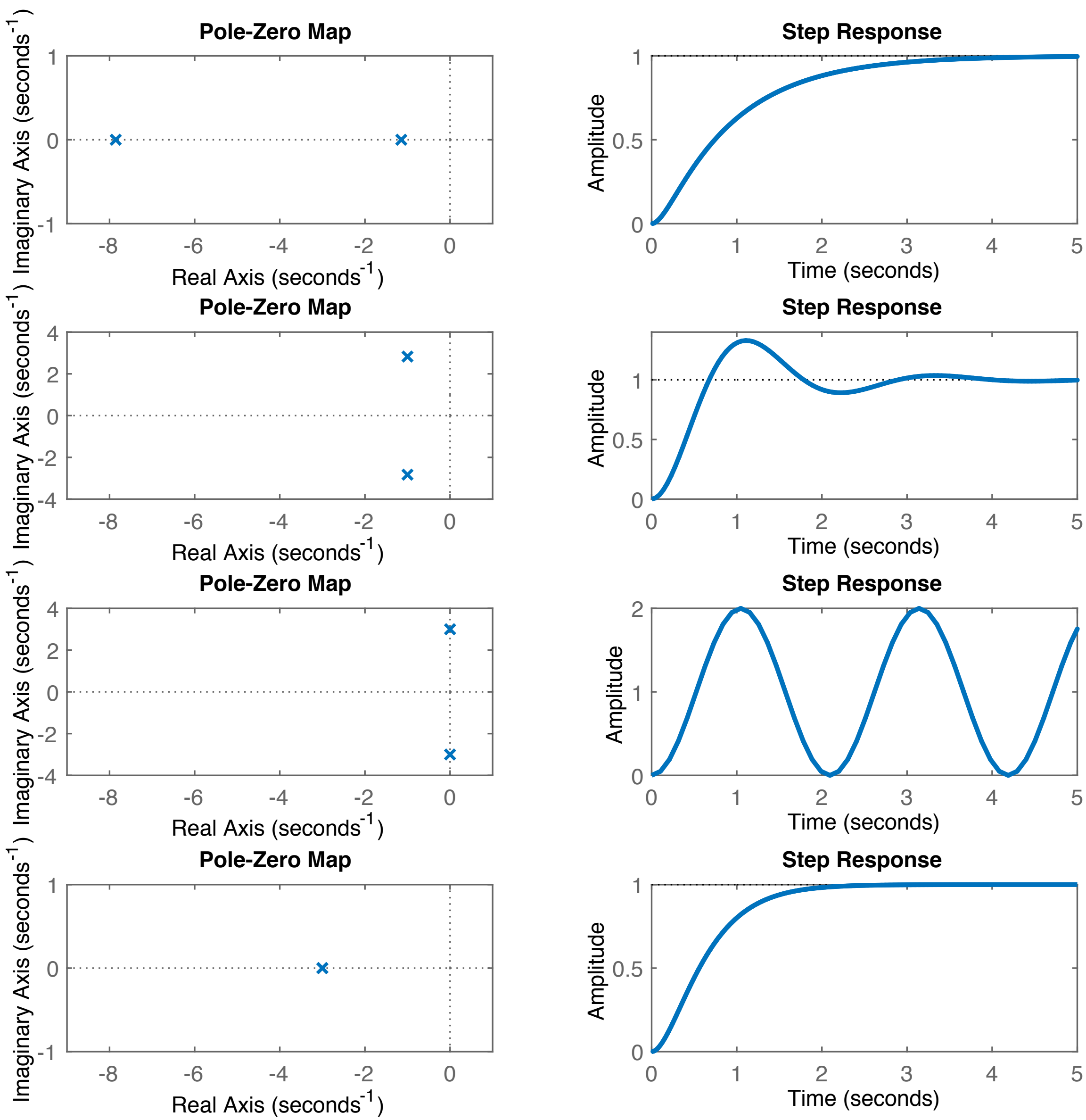
$$\frac{9}{s^2 + 9}$$

```
>> pole(g3)
  0.0000 + 3.0000i
  0.0000 - 3.0000i
```

- Criticamente amortecido: $\zeta = 1$

$$\frac{9}{s^2 + 6s + 9}$$

```
>> pole(g4)
  -3.0000 + 0.0000i
  -3.0000 - 0.0000i
```



$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{b}{s^2 + as + b}$$

Respostas transitórias

Sistemas de 2a-ordem — Resumo:

$$R(s) = \frac{1}{s} \rightarrow \underbrace{\frac{b}{s^2 + as + b}}_{FTMF(s)} \rightarrow Y(s)$$

- Super amortecido: $\zeta > 1$

$$\frac{9}{s^2 + 9s + 9}$$

```
>> pole(g1)
  -7.8541
  -1.1459
```

- **Subamortecido**: $0 < \zeta < 1$

$$\frac{9}{s^2 + 2s + 9}$$

```
>> pole(g2)
 -1.0000 + 2.8284i
 -1.0000 - 2.8284i
```

$$y(t) = 1 - \exp(-\zeta \omega_n t) \cdot \left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t) \right]$$

$$\text{pólos em: } s = \sigma \pm j\omega_d \quad \begin{cases} \sigma = \omega_n \zeta; \\ \omega_d = \omega_n \sqrt{1-\zeta^2}; \end{cases}$$

- Oscilatório: $\zeta = 0$

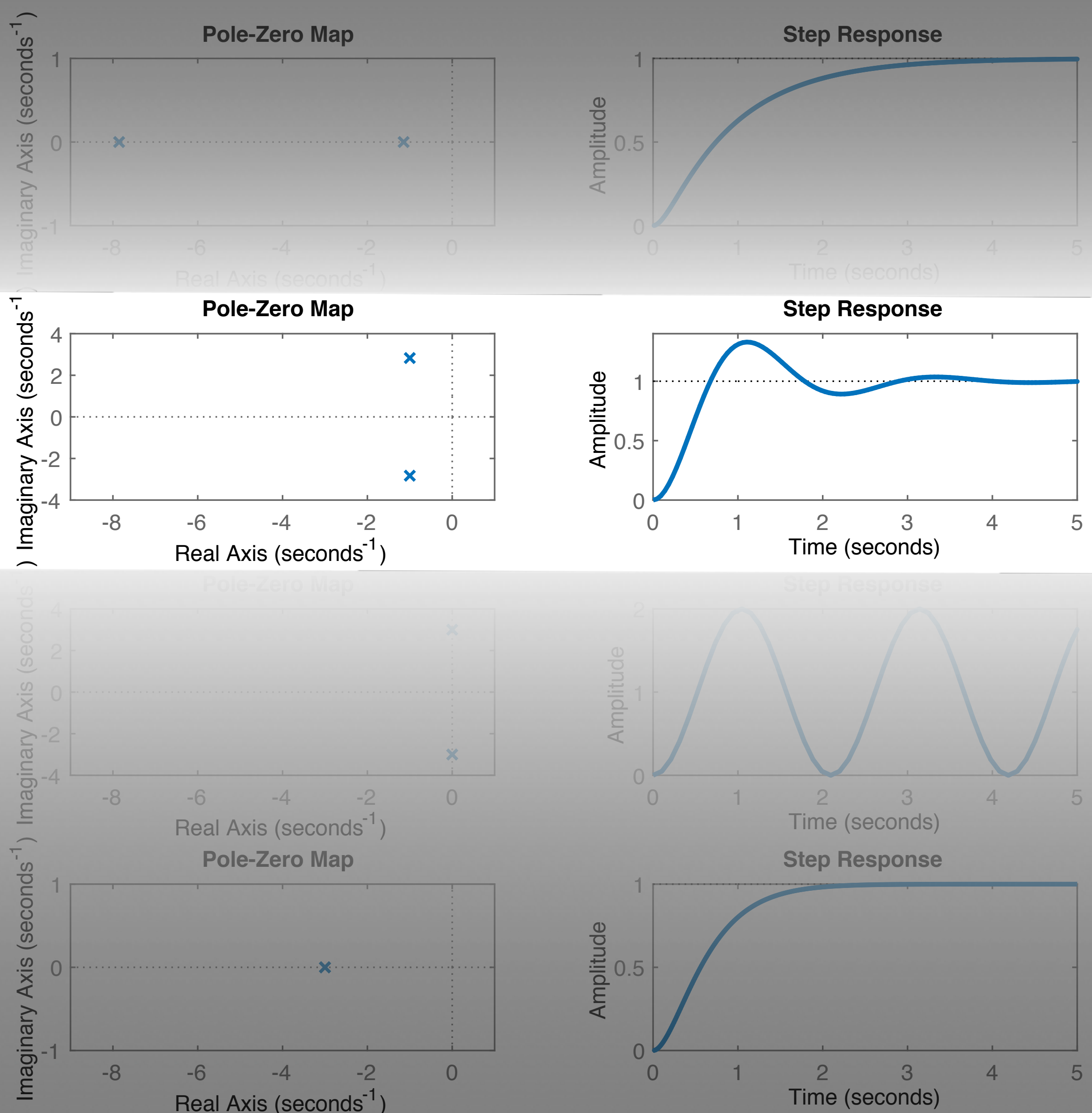
$$\frac{9}{s^2 + 9}$$

```
>> pole(g3)
  0.0000 + 3.0000i
  0.0000 - 3.0000i
```

- Criticamente amortecido: $\zeta = 1$

$$\frac{9}{s^2 + 6s + 9}$$

```
>> pole(g4)
 -3.0000 + 0.0000i
 -3.0000 - 0.0000i
```



$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{b}{s^2 + as + b}$$

Sistemas de 2ª ordem — Detalhes

$$FTMF(s) = \frac{b}{s^2 + as + b} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

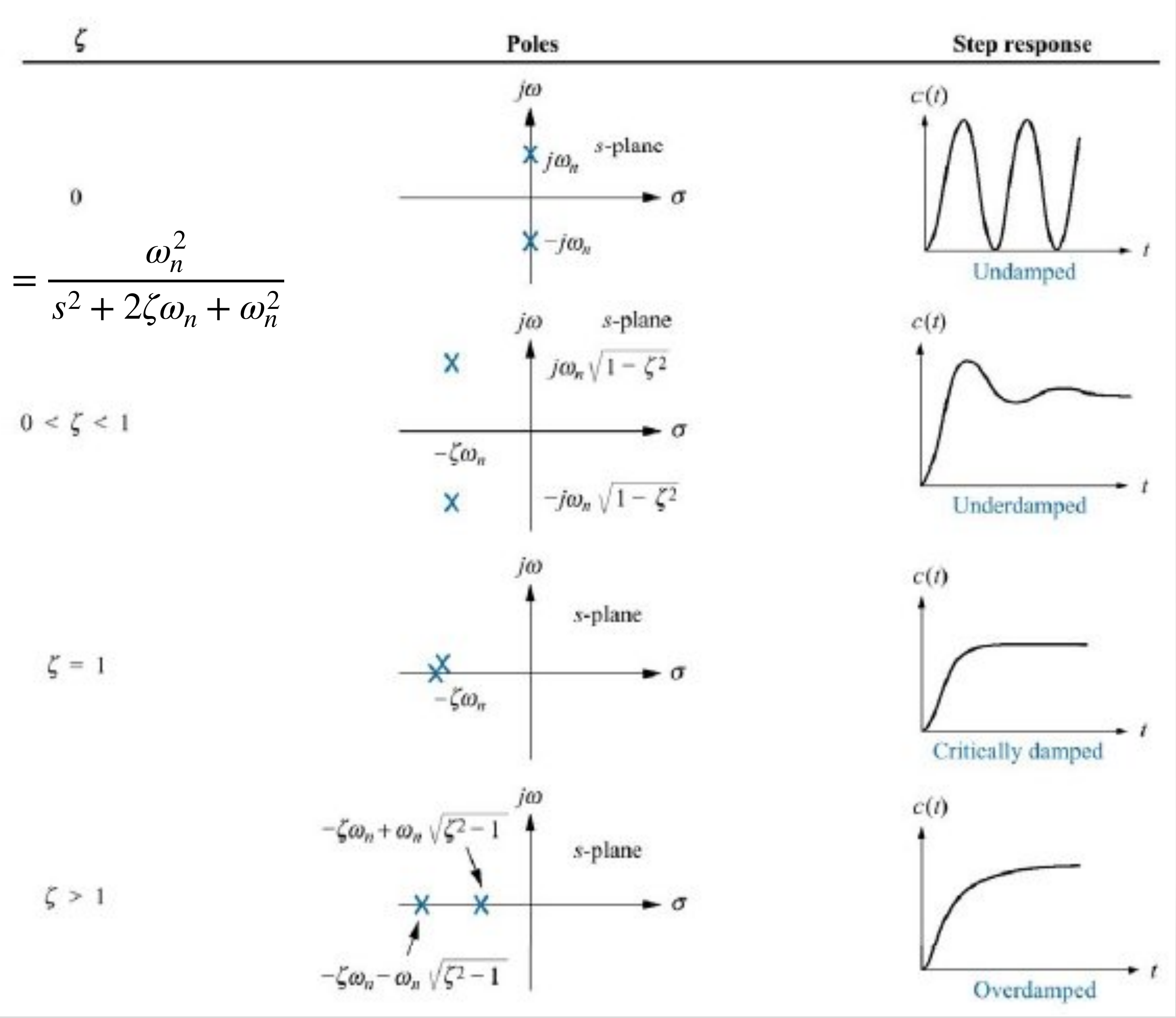
- Frequência natural de oscilação, ω_n: freqüência de oscilação do sistema sem amortecimento ➠ polos puramente imaginários: $a = 0$, polos sobre $j\omega$ em: $\pm j\sqrt{b}$; $\omega_n = \sqrt{b}$; $b = \omega_n^2$.

- Coeficiente de amortecimento, ζ: os polos complexos contêm parte real, σ igual a $-a/2$. A magnitude deste coeficiente modula o decaimento exponencial:

$$\zeta = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n}; \quad a = 2\zeta\omega_n$$

Sistemas de 2ª ordem — Detalhes

$$FTMF(s) = \frac{b}{s^2 + as + b} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$



Sistema subamortecido

- Resposta ao degrau:

$$C(s)=R(s)\,Y(s)=\frac{K}{s(s+p_1)(s+p_2)}=k\,\frac{w_n^2}{s(s^2+2\zeta\,w_n+w_n^2)}=\frac{K_1}{s}+\frac{K_2 s+K_3}{s^2+2\zeta\,w_n^2+w_n^2}$$

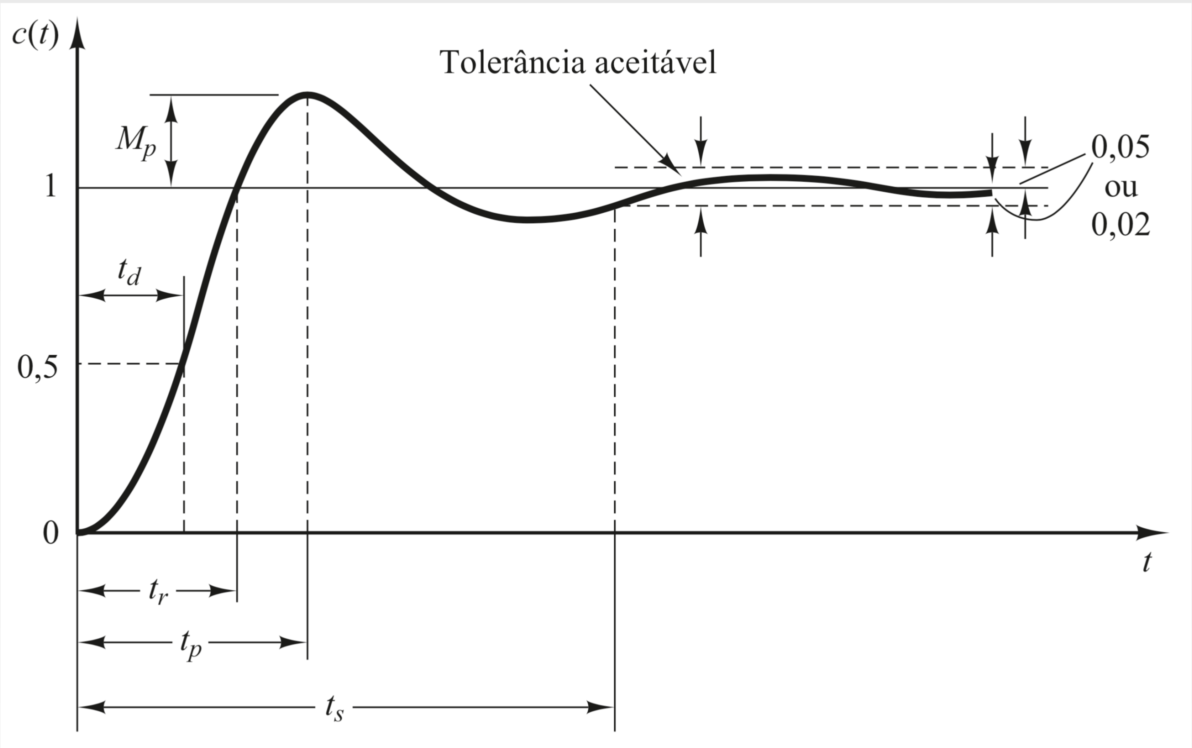
- Assumindo: $0 < \zeta < 1$ (sistema subamortecido):

$$C(s)=\frac{1}{s}-\frac{(s+\zeta w_n)+\dfrac{\zeta}{\sqrt{1-\zeta^2}}\,w_n\sqrt{1-\zeta^2}}{(s+\zeta w_n)^2+w_n(1-\zeta^2)}$$

$$c(t)=1-e^{-\zeta w_n t}\left(\cos w_n\sqrt{1-\zeta^2}\,t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin w_n\sqrt{1-\zeta^2}\,t\right)$$

$$=1-\frac{1}{\sqrt{1-\zeta^2}}\,e^{-\zeta w_n t}\cos\left(w_n\sqrt{1-\zeta^2}\,t-\phi\right)\quad \text{where } \phi=\tan^{-1}\left(\zeta/\sqrt{1-\zeta^2}\right)$$

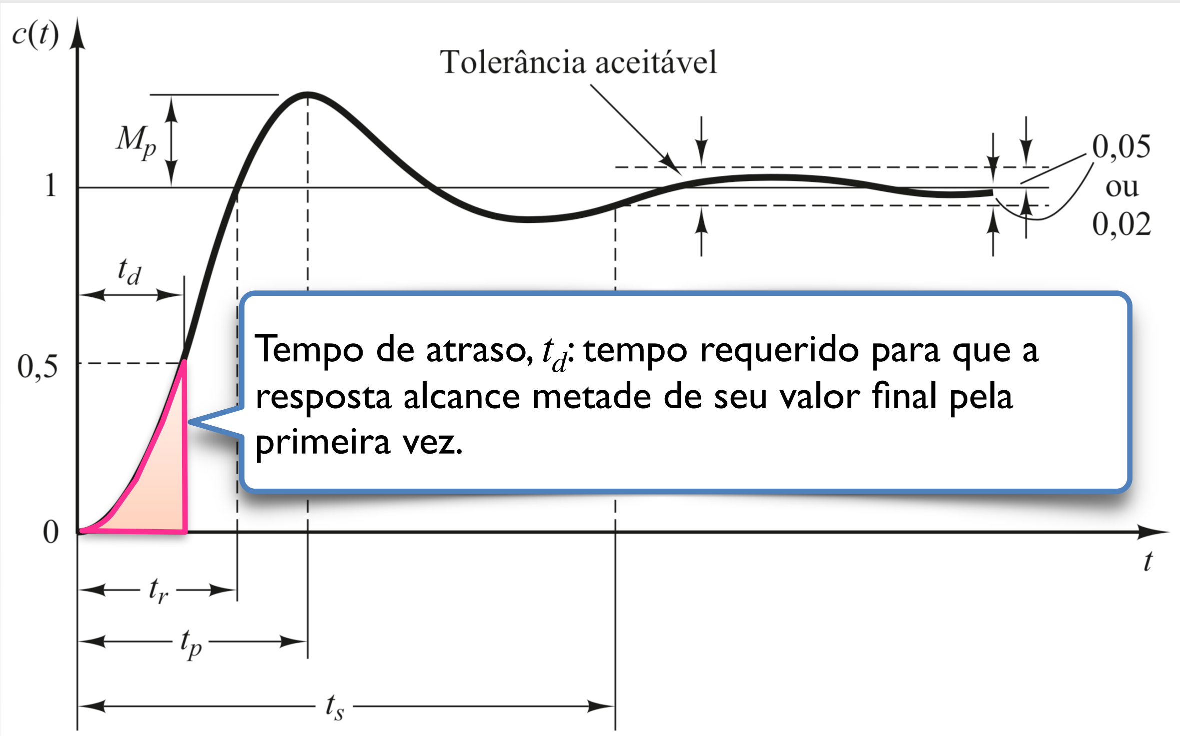
Respostas de Sistemas de 2ª ordem…



Parâmetros temporais:

1. Tempo de subida, t_r;
2. Tempo de pico, t_p;
3. Máximo sobressinal, M_p, $\%\,OS$ (*overshoot*);
4. Tempo de acomodação, t_s, (*settling time*);
5. Tempo de atraso, t_d.

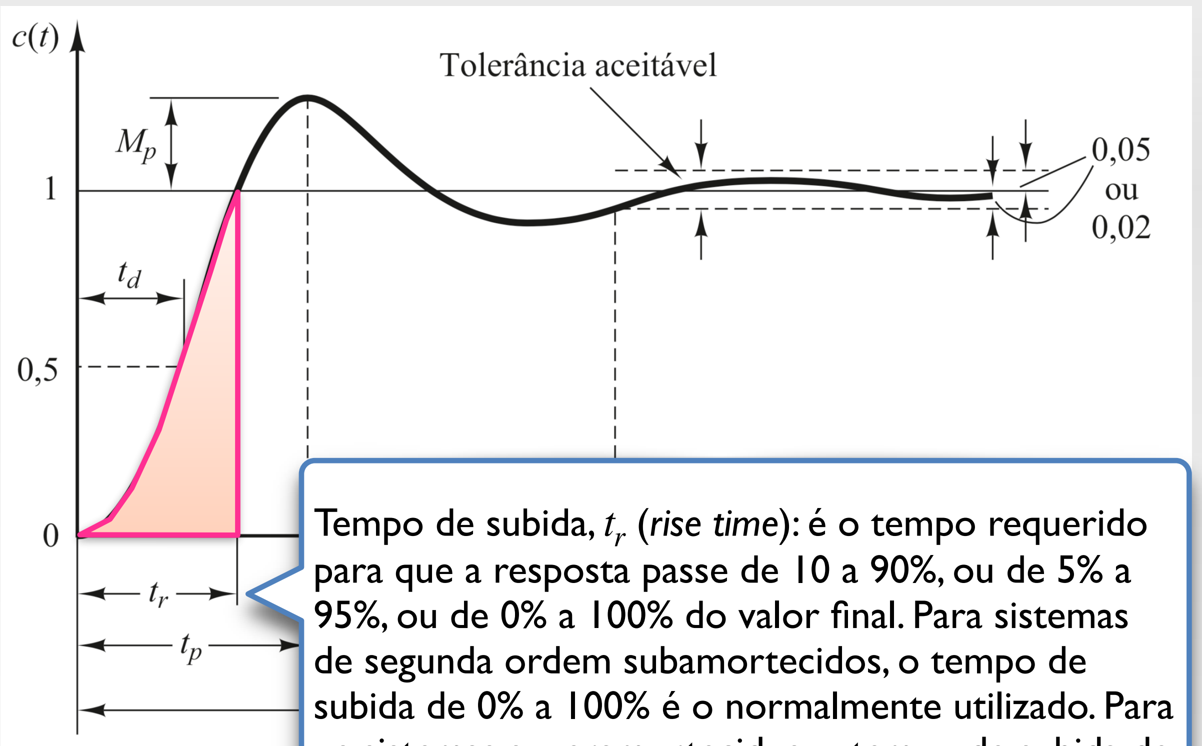
Respostas de Sistemas de 2ª ordem...



Parâmetros temporais:

1. Tempo de subida, t_r;
2. Tempo de pico, t_p;
3. Máximo sobressinal, M_p,
   $\% OS$ (*overshoot*);
4. Tempo de acomodação, t_s, (*settling time*);
5. Tempo de atraso, t_d.

Respostas de Sistemas de 2ª ordem...

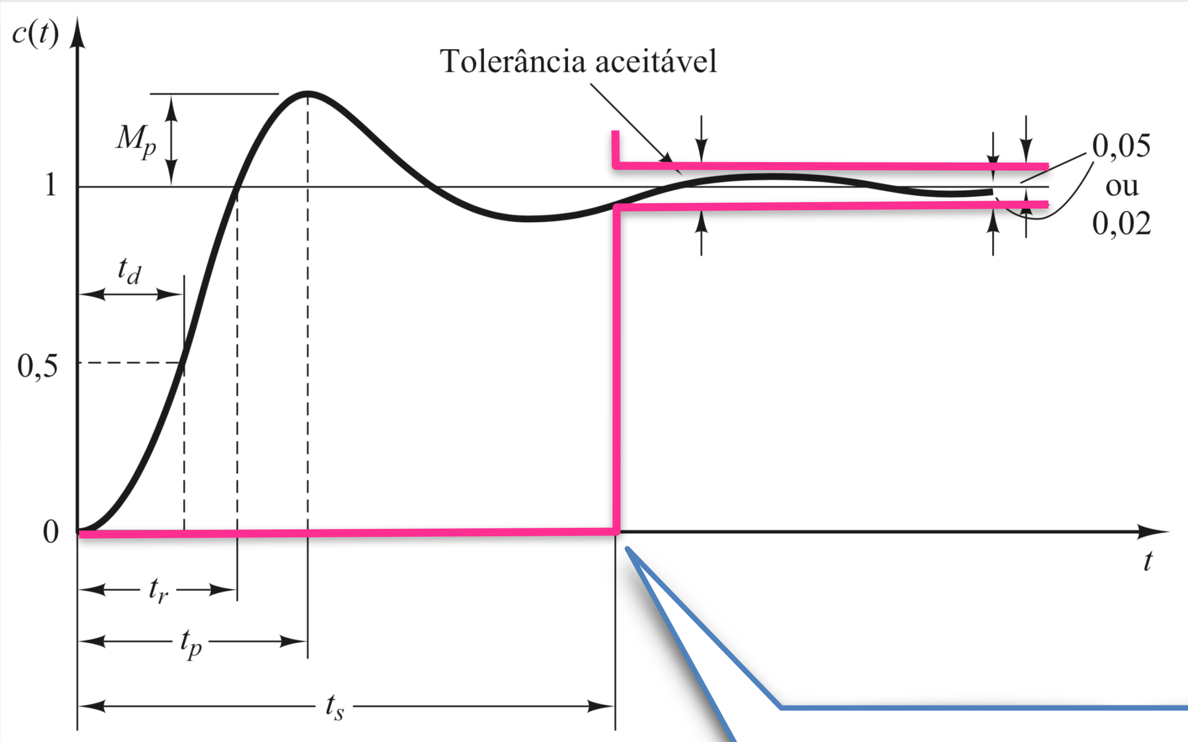


Tempo de subida, t_r (*rise time*): é o tempo requerido para que a resposta passe de 10 a 90%, ou de 5% a 95%, ou de 0% a 100% do valor final. Para sistemas de segunda ordem subamortecidos, o tempo de subida de 0% a 100% é o normalmente utilizado. Para os sistemas superamortecidos, o tempo de subida de 10% a 90% é o mais comumente utilizado.

Parâmetros temporais:

1. Tempo de subida, t_r;
2. Tempo de pico, t_p;
3. Máximo sobressinal, M_p, $\% OS$ (*overshoot*);
4. Tempo de acomodação, t_s, (*settling time*);
5. Tempo de atraso, t_d.

Respostas de Sistemas de 2ª ordem...

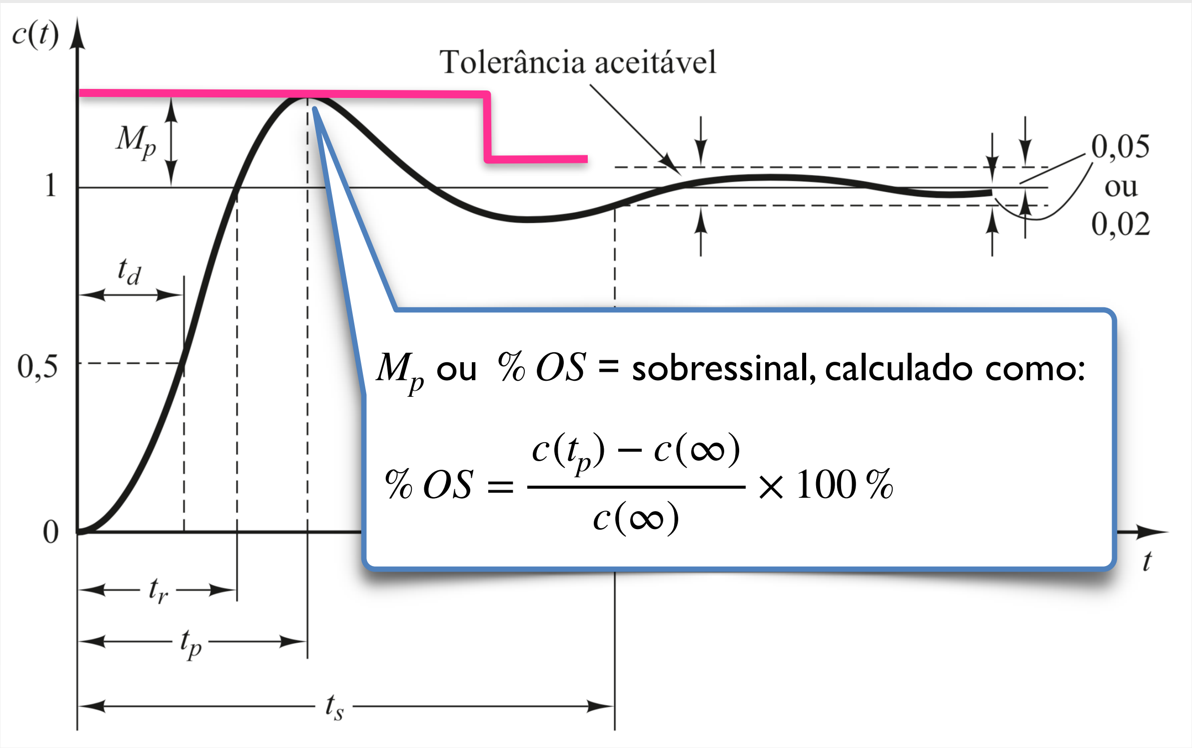


Parâmetros temporais:

1. Tempo de subida, t_r;
2. Tempo de pico, t_p;
3. Máximo sobressinal, M_p, $\%\,OS$ (*overshoot*);
4. Tempo de acomodação, t_s, (*settling time*);
5. Tempo de atraso, t_d.

Tempo de acomodação, t_s (*settling time*): é o tempo necessário para que a curva de resposta alcance valores em uma faixa (geralmente de 2% ou 5%) em torno do valor final, aí permanecendo indefinidamente. O tempo de acomodação está relacionado à maior constante de tempo do sistema de controle.

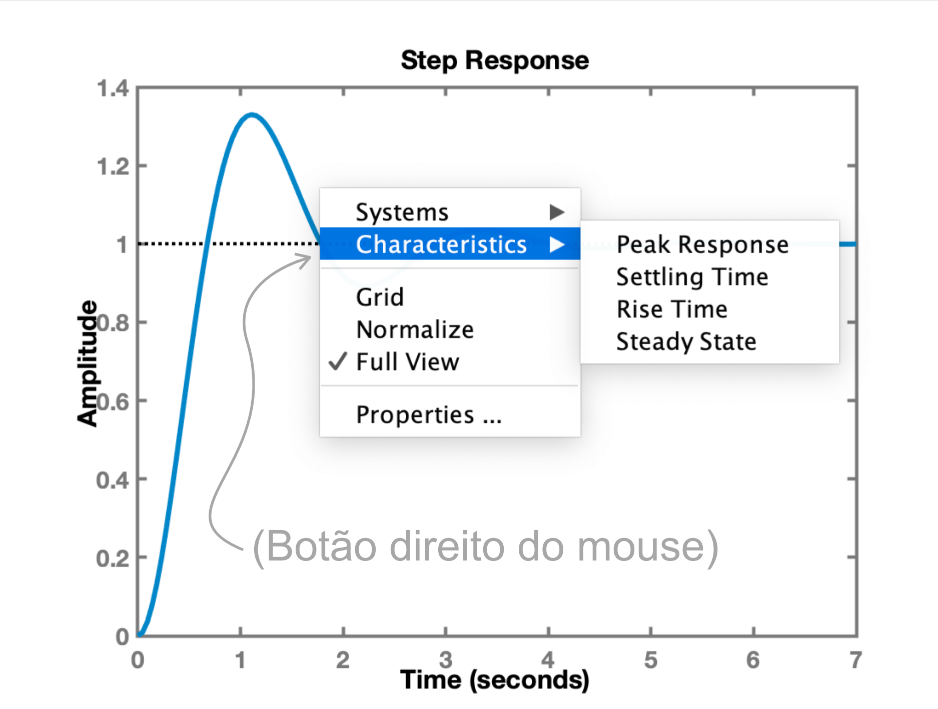
Respostas de Sistemas de 2ª ordem…



Parâmetros temporais:

1. Tempo de subida, t_r;
2. Tempo de pico, t_p;
3. Máximo sobressinal, M_p, $\% OS$ (*overshoot*);
4. Tempo de acomodação, t_s, (*settling time*);
5. Tempo de atraso, t_d.

Respostas de Sistemas de 2ª ordem...



Parâmetros temporais:

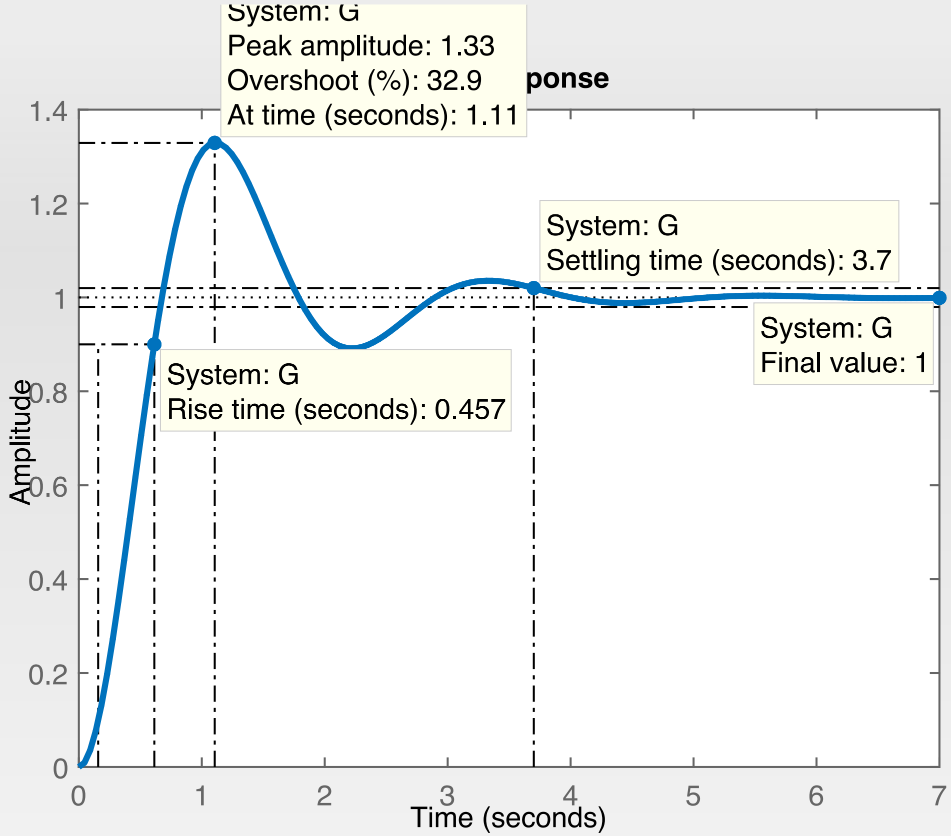
1. Tempo de subida, t_r;
2. Tempo de pico, t_p;
3. Máximo sobressinal, M_p, $\% \, OS$ (*overshoot*);
4. Tempo de acomodação, t_s, (*settling time*);
5. Tempo de atraso, t_d.

Matlab:

```
>> G=tf(9, [1 2 9]);
>> zpk(G)

        9
  --------------
  (s^2 + 2s + 9)

Continuous-time zero/pole/gain model.

>> step(G)
```

```
>> stepinfo(G)
ans =
  struct with fields:

        RiseTime: 0.4568
    SettlingTime: 3.7005
     SettlingMin: 0.8916
     SettlingMax: 1.3293
       Overshoot: 32.9277
      Undershoot: 0
            Peak: 1.3293
        PeakTime: 1.1052
>>
```

Respostas de Sistemas de 2ª ordem…



Parâmetros temporais:

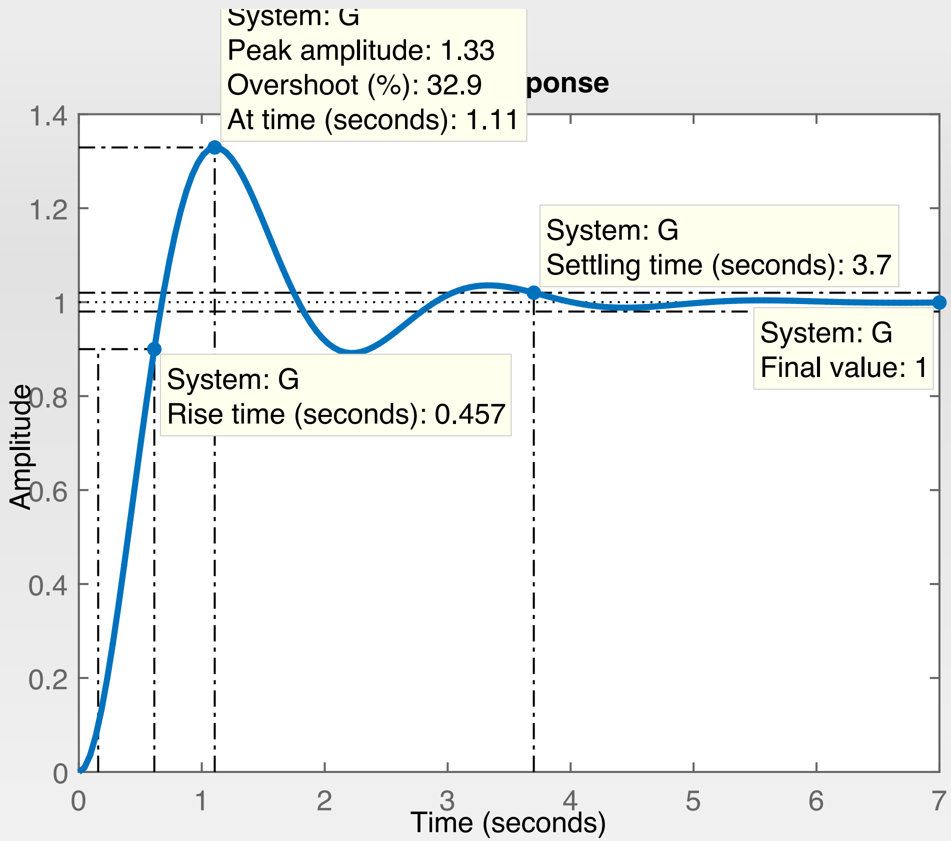
1. Tempo de subida, t_r;
2. Tempo de pico, t_p;
3. Máximo sobressinal, M_p,
   $\% OS$ (*overshoot*);
4. Tempo de acomodação, t_s, (*settling time*);
5. Tempo de atraso, t_d.

Matlab:

```
>> G=tf(9, [1 2 9]);
>> zpk(G)

         9
  --------------
  (s^2 + 2s + 9)

Continuous-time zero/pole/gain model.

>> step(C)
```

```
>> stepinfo(C)
ans =
  struct with fields:

        RiseTime: 0.4568
    SettlingTime: 3.7005
     SettlingMin: 0.8916
     SettlingMax: 1.3293
       Overshoot: 32.9277
      Undershoot: 0
            Peak: 1.3293
        PeakTime: 1.1052
>>
```

Respostas de Sistemas de 2ª ordem subamortecidos ($0 < \zeta < 1$)



Tempo de pico, t_p (*peak time*): se obtêm resolvendo $\dfrac{\partial y}{\partial t} = 0$:

$$0 = \frac{dy}{dt} = L^{-1}\{\dot{c}(t)\} = L^{-1}\left\{s \cdot \frac{1}{s} \cdot \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}\right\}$$

$$\dot{c}(t) = \frac{w_n}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} \sin(w_n \sqrt{1-\zeta^2}\, t)$$

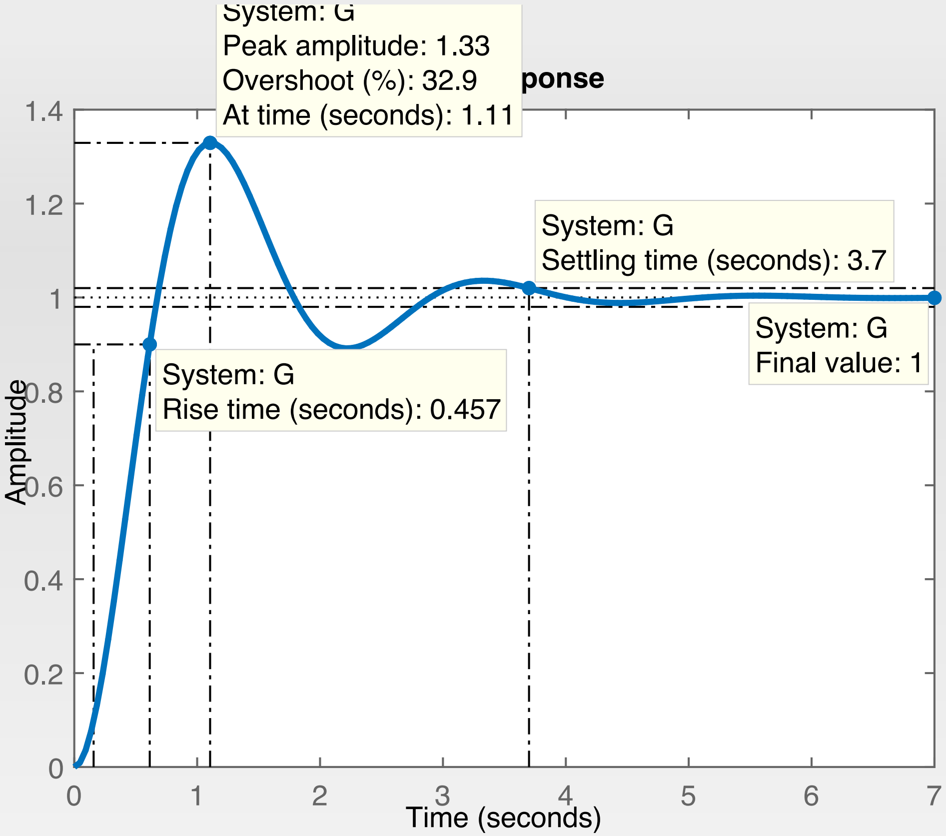
$$w_n \sqrt{1-\zeta^2}\, t = n\pi$$

$n = 0$: primeiro ponto da curva de resposta (inclinação zero);

$n = 1$: primeiro pico, equivale a $t = t_p$:

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Respostas de Sistemas de 2ª ordem subamortecidos $(0 < \zeta < 1)$



Percentual de overshoot (sobressinal), $\% OS$:

$$\%OS = \frac{c_{max} - c_{final}}{c_{final}} \times 100\%$$

$c_{max} = c(t_p)$; uso da equação anterior sobre:

$$c(t) = 1 - e^{-\zeta w_n t}\left(\cos\left(w_n\sqrt{1-\zeta^2}\,t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(w_n\sqrt{1-\zeta^2}\,t\right)\right)$$

$$c_{max} = c(t_p) = 1 - e^{-(\zeta\pi/\sqrt{1-\zeta^2})}\left(\cos(\pi) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\pi)\right)$$

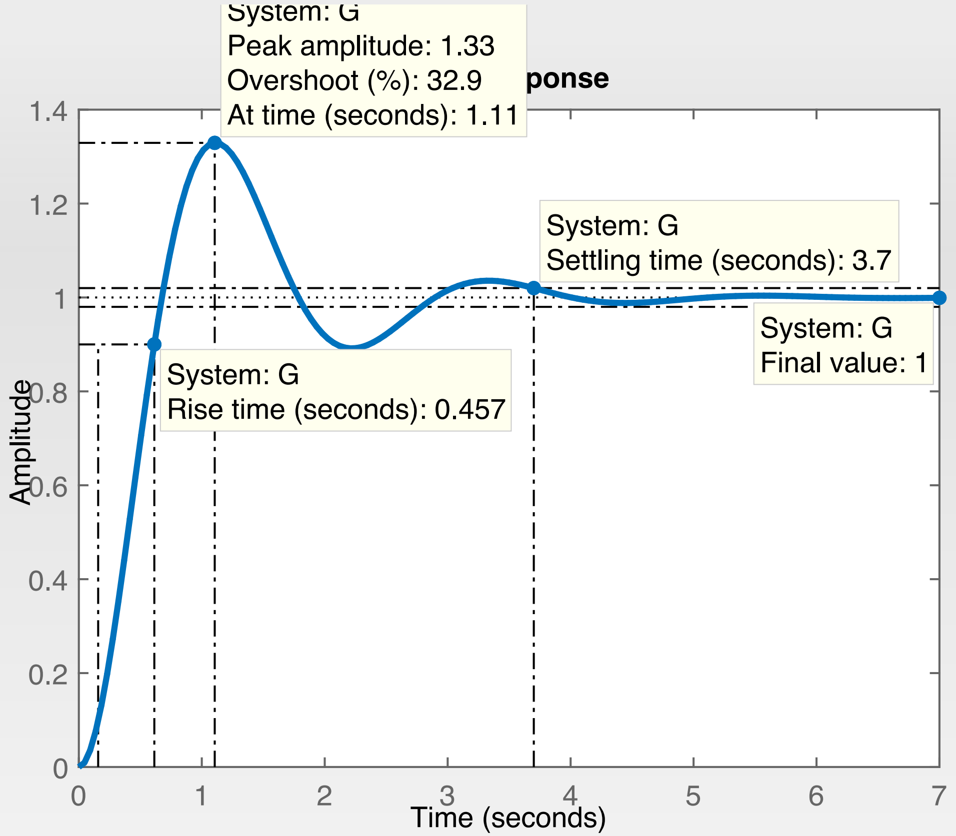
$$c_{max} = 1 + e^{-(\zeta\pi/\sqrt{1-\zeta^2})}$$

$$c_{final} = 1$$

$$\% OS = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100\,\%$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Respostas de Sistemas de 2ª ordem subamortecidos ($0 < \zeta < 1$)



Tempo de assentamento (settling time), t_s ($\pm 2\,\%$ de tolerância):

A partir de:

$$c(t) = 1 - e^{-\zeta w_n t}\left(\cos\left(w_n\sqrt{1-\zeta^2}\,t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(w_n\sqrt{1-\zeta^2}\,t\right)\right)$$

Isolamos:

$$e^{-\zeta w_n t}\frac{1}{\sqrt{1-\zeta^2}} = 0{,}02 \qquad (4.40)$$

Sabe-se ainda que em $t = t_p$:

$$\cos\left(w_n\sqrt{1-\zeta^2}\,t - \phi\right) = 1$$

Resolvendo (4.40) para $t = t_p$:

$$t_s = \frac{-\ln\left(0{,}02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

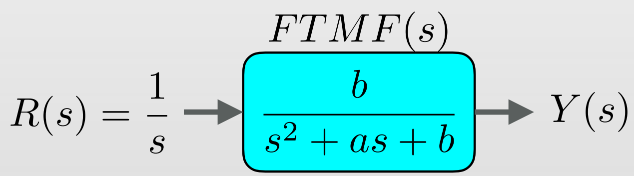
O numerador desta eq. varia entre 3,91 à 4,74 enquanto $0 < \zeta < 1$, então uma aproximação seria: ➠

$$t_s = \frac{4}{\zeta\omega_n}$$

Respostas de Sistemas de 2ª ordem subamortecidos $(0 < \zeta < 1)$ — Resumo

Equações do sistema em MF:

$$FTMF(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{b}{s^2 + as + b}$$

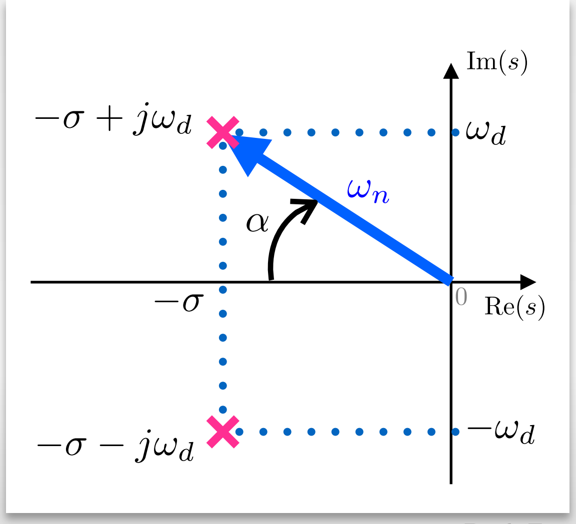


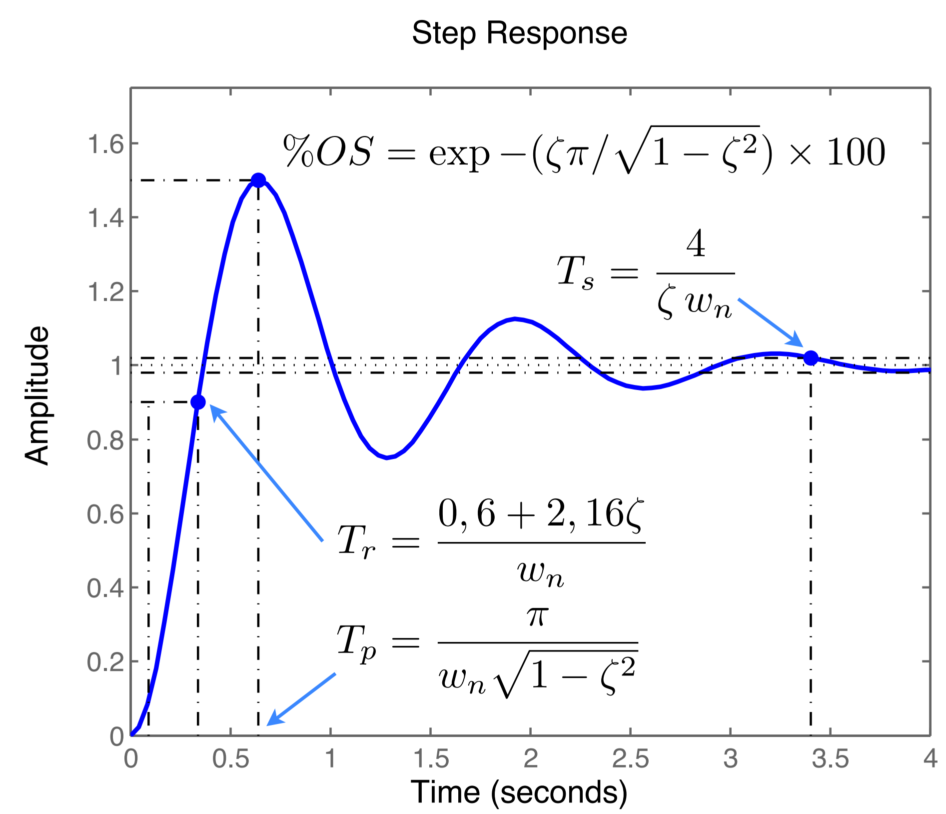
$$\frac{Y(s)}{R(s)} = \frac{K}{(s+p_1)(s+p_2)} = K\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$

pólos em: $s = \sigma \pm j\omega_d$ ou: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$$\begin{aligned} \sigma &= \omega_n \cos(\alpha) = \omega_n \zeta; \\ \omega_d &= \omega_n \sin(\alpha) = \omega_n\sqrt{1-\zeta^2}; \\ \zeta &= \cos(\alpha); \\ \sin(\alpha) &= \sqrt{1-\zeta^2}; \end{aligned}$$

$$0 < \zeta < 1$$





Linhas guia no plano-s

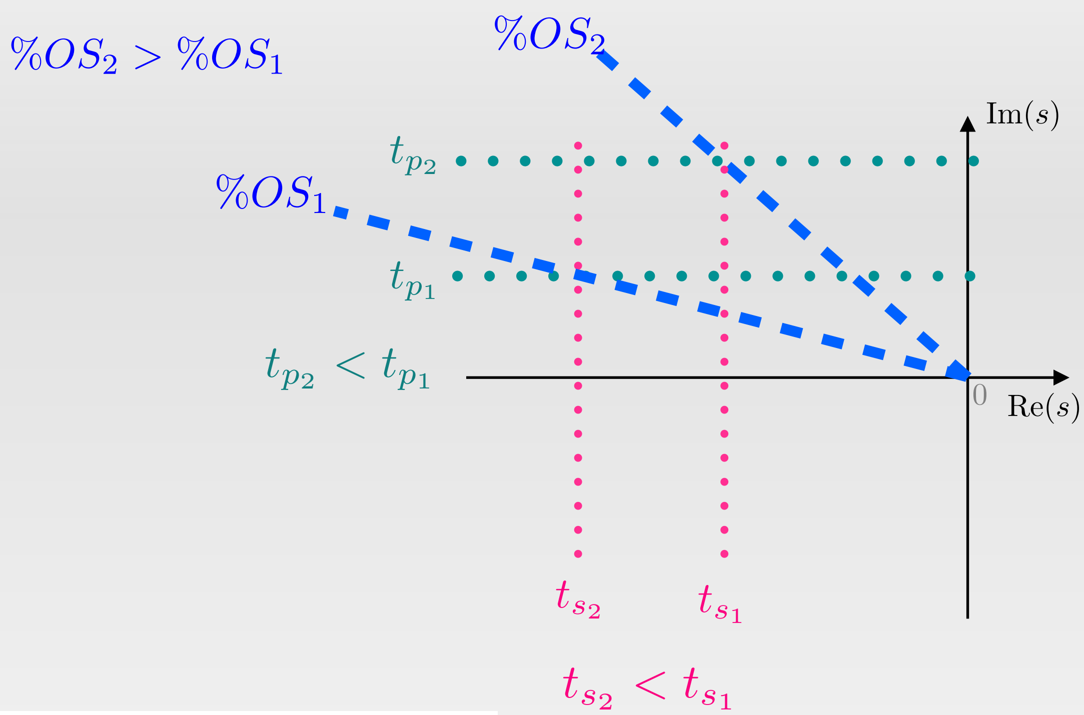
$$\%OS = \exp\left(-\zeta\pi/\sqrt{1-\zeta^2}\right) \times 100$$

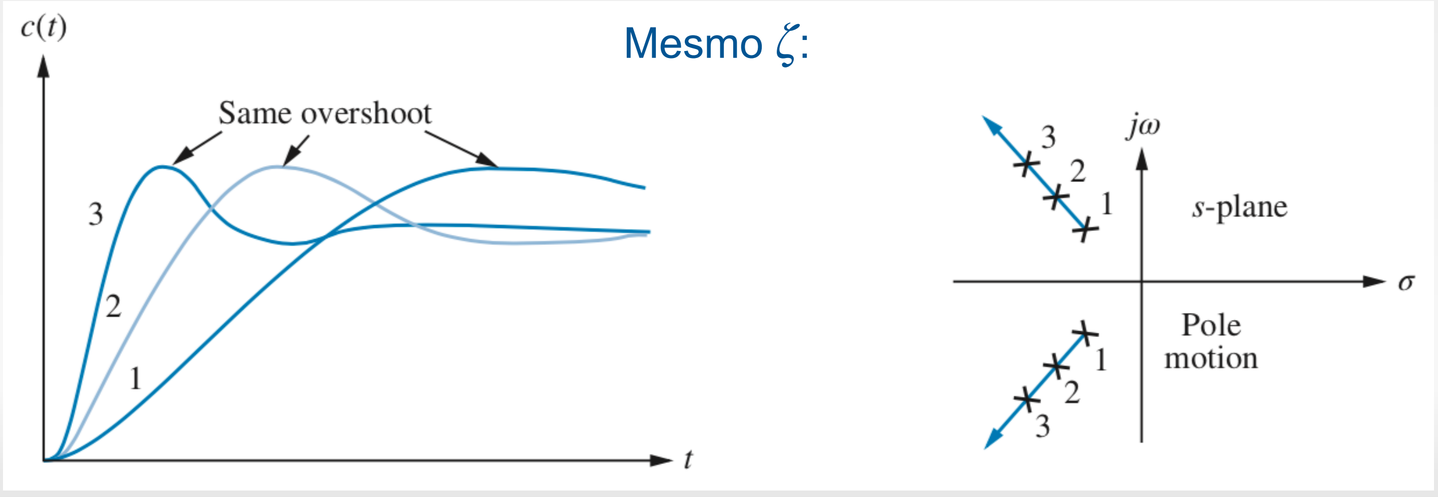
$$\zeta = \cos(\theta)$$

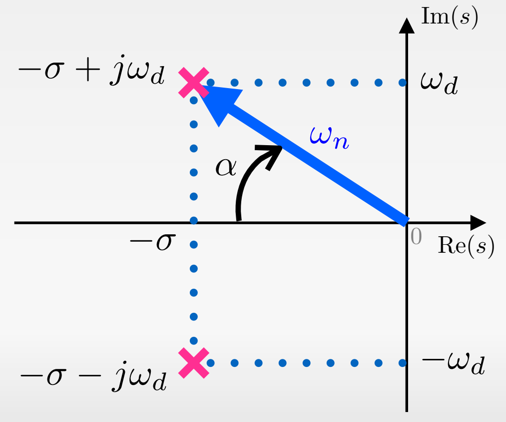
$$\Downarrow$$

Mesmas linhas radiais:

= mesmo ζ







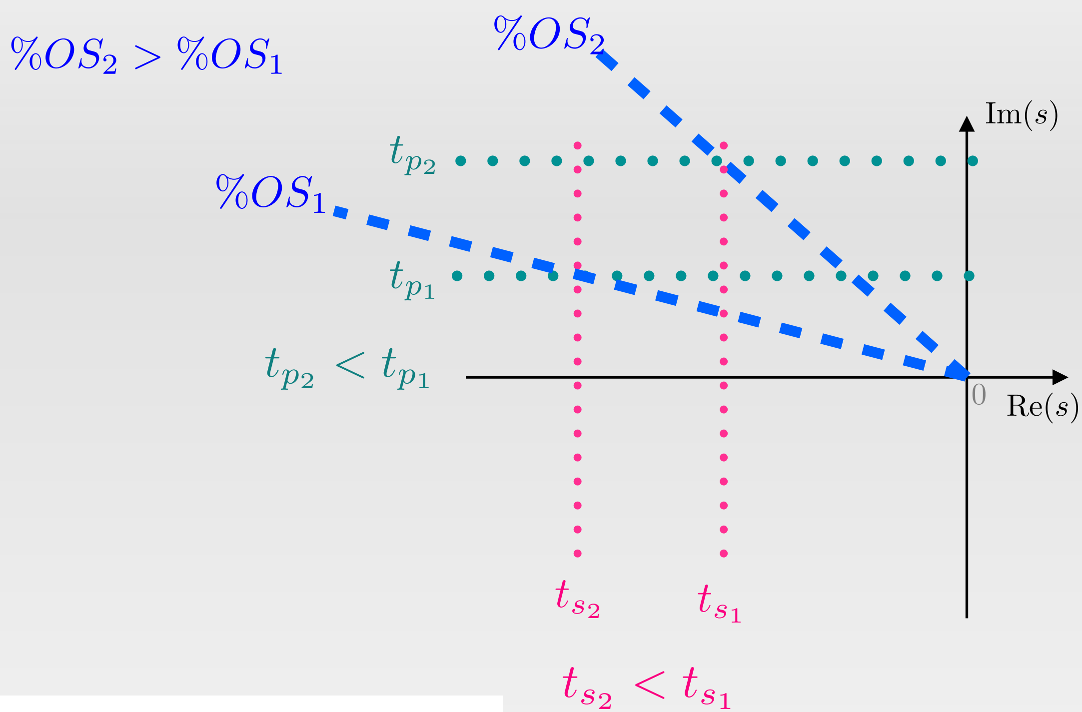
Linhas guia no plano-s

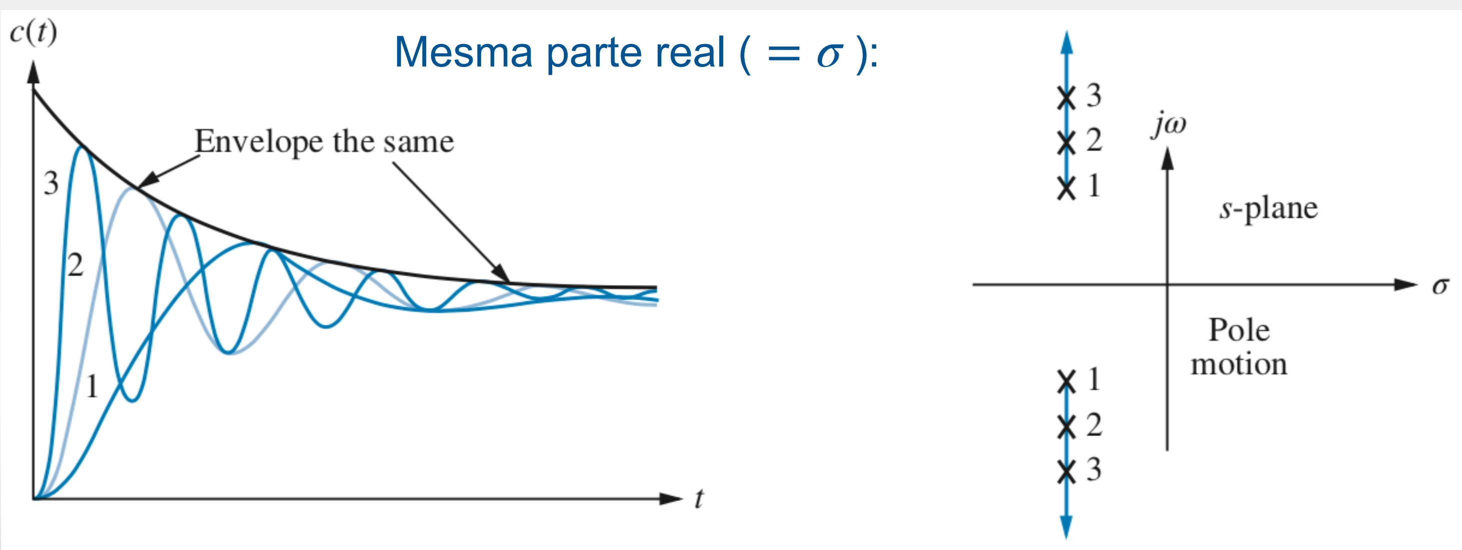
$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

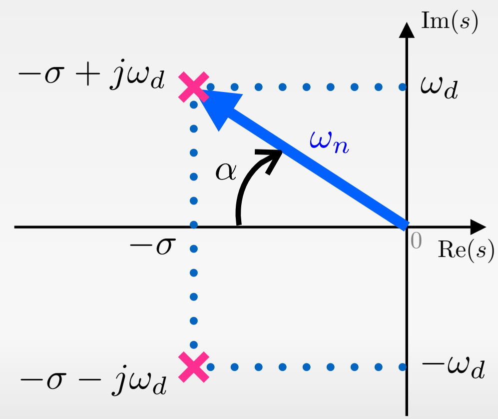
t_p

⇩

Inversamente proporcional à parte imaginária do pólo.







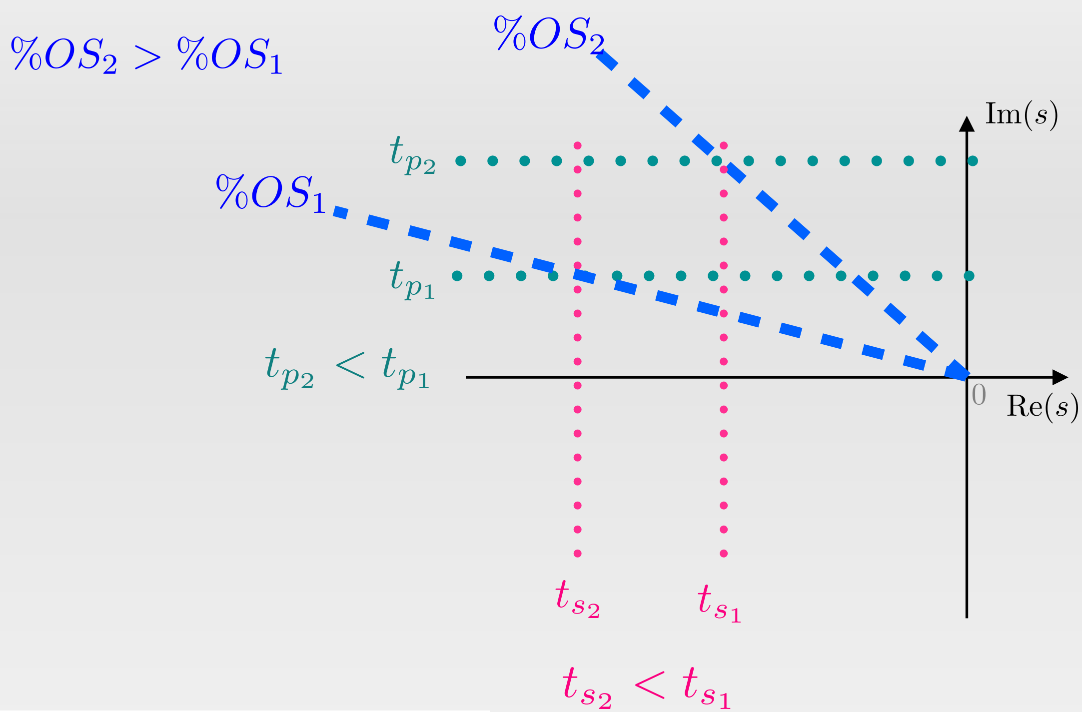
Linhas guia no plano-s

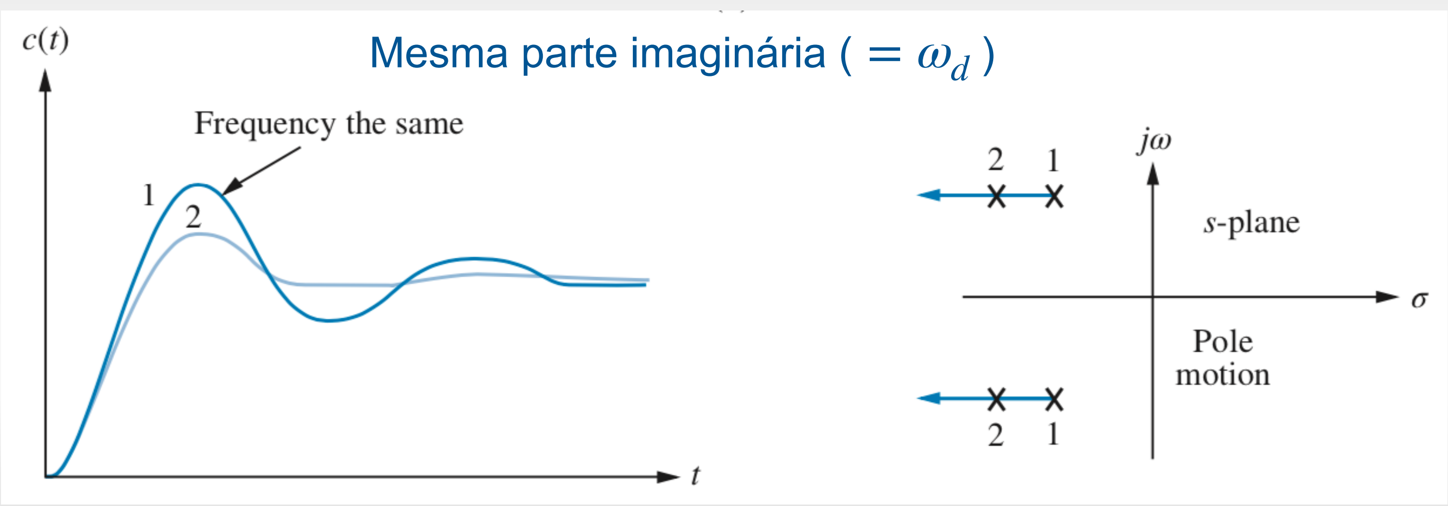
$$t_s = \frac{4}{\zeta\,\omega_n} = \frac{4}{\sigma}$$

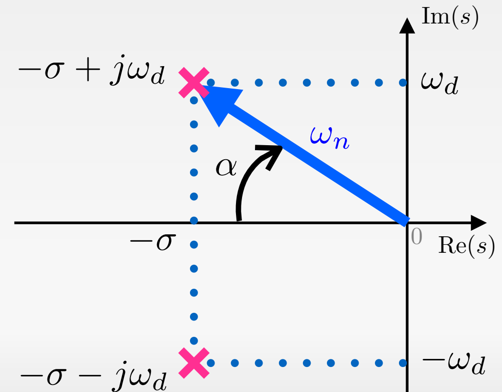
t_s

⇩

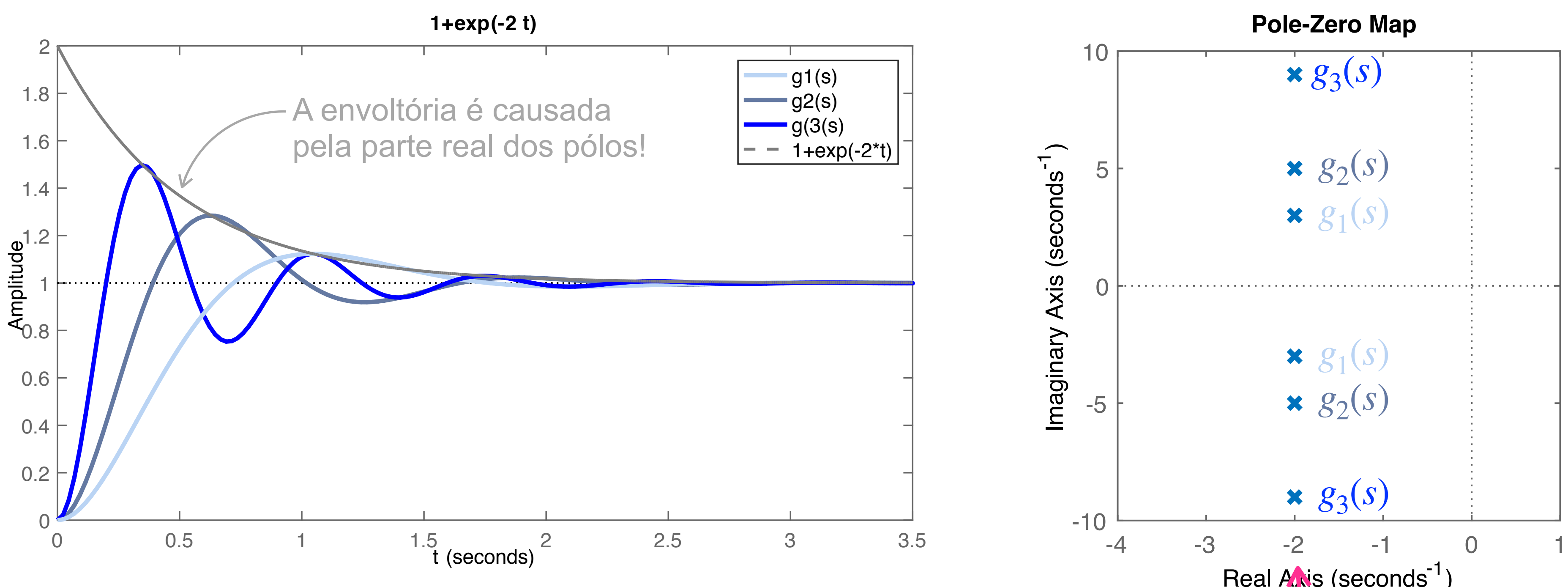
Inversamente proporcional à parte real do pólo







Sistemas subamortecidos (mesmo σ)



$$g_1(s) = \frac{13}{(s^2+4s+13)} = \frac{13}{(s+2+j3)(s+2-j3)}$$
$$= \frac{(3{,}6056)^2}{s^2+2(0{,}5547)(3{,}6056)s+(3{,}6056)^2}$$
$$g_2(s) = \frac{9}{(s^2+2s+9)} = \frac{13}{(s+2+j5)(s+2-j5)}$$
$$= \frac{(5{,}3852)^2}{s^2+2(0{,}3714)(5{,}3852)s+(5{,}3852)^2}$$
$$g_3(s) = \frac{85}{(s^2+4s+85)} = \frac{85}{(s+2+j9)(s+2-j9)}$$
$$= \frac{(9.2195)^2}{s^2+2(0{,}2169)(9.2195)s+(9.2195)^2}$$

Mesma parte real

Resumo sistemas subamortecidos ($0 < \zeta < 1$)

$$FTMF(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{b}{s^2 + as + b}$$

$$y(t) = 1 - \exp(-\zeta\omega_n t) \cdot \left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t)\right]$$

pólos em: $s = \sigma \pm j\omega_d$ ou: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$$\%OS = \exp\left(-\zeta\pi/\sqrt{1-\zeta^2}\right) \times 100$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

$$\zeta = \cos(\alpha)$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$t_s = \frac{-\ln\left(0,02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \quad \text{para: } 0 < \zeta < 0,9$$

$$t_r = \frac{0,6 + 2,16\,\zeta}{\omega_n}$$